The stabilization of the Frobenius-Hecke traces on the intersection cohomology of orthogonal Shimura varieties

The stabilization of the Frobenius-Hecke traces on the intersection cohomology of orthogonal Shimura varieties

A dissertation presented

by

Yihang Zhu

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

April 2017

Dissertation Advisor: Professor Mark Kisin Yihang Zhu

The stabilization of the Frobenius-Hecke traces on the intersection cohomology of orthogonal Shimura varieties

Abstract

The orthogonal Shimura varieties are associated to special orthogonal groups over $\mathbb{Q}$ of signature $(n, 2)$ at infinity. In this work we prove a version of Morel's formula for the Frobenius-Hecke traces on the intersection cohomology of their Baily-Borel compactifications. We then stabilize the formula, in terms of Kottwitz's simplified expression for the geometric side of the Arthur-Selberg trace formula for test functions that are stable cuspidal at infinity.

Contents

Dedicated to Han Wu,

for her support.

Acknowledgements

First and foremost I would like to thank my advisor Mark Kisin, who introduced me to the subject of Shimura varieties, as a gateway towards the fantastic world of Langlands program and number theory in general. I thank him for suggesting this particular project to me, which turned out to be full of thrill, excitement, and beauty. During my graduate study, I have had the privilege of receiving his constant guidance, encouragement, support, pressure, kindness, humor, and friendship. Moreover, I learned from him the way to do mathematics and the meaning of being a mathematician.

Equally I thank Tasho Kaletha, for generously spending numerous hours explaining to me the subject of endoscopy and many other things, and for his detailedly written emails answering my questions, during a crucial stage of this work.

It should be clear that the beautiful and pioneering work of Sophie Morel on weighted complexes and non-compact Shimura varieties is the inspiration and model of this work. I thank her for conversations that were greatly encouraging and beneficial to me. I also thank her for reading through the draft carefully and pointing out several misstatements and typos. Needless to say, any remaining errors or inaccuracies are my own.

It is also my great pleasure to thank Robert Kottwitz for kindly sending his notes on the stabilization of the topological Lefschetz formula to me. These notes have been the ultimate inspiration for the stabilization process here, even though during the main stage of this project I was not in hold of them and only learned about them through Morel's work.

At the very beginning of this project, Keerthi Madapusi Pera informed me of the important papers to study in order to enter the subject of non-compact Shimura varieties. I thank him for that.

I would also like to thank George Boxer, Kai-Wen Lan, Chao Li, Koji Shimizu, Sug Woo Shin, Olivier Taïbi, David Vogan, Zhiwei Yun, and Rong Zhou for informative conversations and for answering my questions patiently, during various stages of this work.

Finally, I would like to thank the members and staff of the Harvard Mathematics Department for their support and companionship during my graduate study.

Introduction

0.1 Motivation

Initiated by the work of Eichler, Shimura, Kuga, Sato, and Ihara, among others, the problem of determining the Hasse-Weil zeta functions of Shimura varieties in terms of automorphic L-functions has received continual study, and it has been one of the central problems in the Langlands program. The so-called Langlands-Kottwitz method to tackle this problem seeks to express Frobenius-Hecke traces on the cohomology of Shimura varieties in terms of the stable Arthur-Selberg trace formula. Kottwitz [40] formulates this conjectural expression precisely, and it is this conjecture that we confirm for orthogonal Shimura varieties in this work.

Before the more precise discussion, we mention that there are at least two main motivations for studying the Hasse-Weil zeta functions of Shimura varieties. Firstly, the structures of these functions, or more generally the structures of the étale cohomology of Shimura varieties, reflect the reciprocity between automorphic representations and Galois representations (or more generally motives) predicted by Langlands. Understanding of such structures would serve as a key step in establishing cases of global Langlands correspondences. Secondly, it is conjectured in general that Hasse-Weil zeta functions of algebraic varieties over number fields should satisfy good analytic properties, such as meromorphic extension and functional equation. The only known way to prove this conjecture is to relate the zeta function in question to automorphic L-functions. Apart from essentially zero dimensional examples, the only known cases fall into the categories of either abelian varieties or Shimura varieties.

To calculate the Hasse-Weil zeta functions of general Shimura varieties, the original method used by Eichler and Shimura in the one dimensional case, that of congruence relations, turns out to be insufficient for generalization. Langlands [49][50][51] and Kottwitz [40][41] have developed the strategy of comparing the Grothendieck-Lefschetz trace formula for Shimura varieties and the stable Arthur-Selberg trace formula. In the following we recall the precise conjectures of Kottwitz [40].

0.2 Precise conjectures

Let (G, X) be a Shimura datum of reflex field E. For a neat open compact subgroup $K \subset G(\mathbb{A}_f)$, we obtain the canonical model

$$\mathrm{Sh}_K = \mathrm{Sh}_K(G, X),$$

which is a smooth quasi-projective algebraic variety over E. Consider the Baily-Borel compactification (also known as minimal compactification)

$$\overline{\mathrm{Sh}_K} = \overline{\mathrm{Sh}_K(G, X)}$$

of $\mathrm{Sh}_K(G, X)$. This is a normal projective variety over E containing Sh_K as an open subvariety. Let

$$\mathbf{IH}^* = \mathbf{IH}^*(\overline{\mathrm{Sh}_K})$$

be (the formal alternating direct sum of) the (étale) intersection cohomology of $\overline{\mathrm{Sh}_K} \times \mathrm{Spec}\, \bar{E}$ with coefficients in an automorphic λ-adic sheaf. More precisely,

$$\mathbf{IH}^* = \mathbf{H}^*_{\text{ét}}(\overline{\mathrm{Sh}_K} \times \mathrm{Spec}\, \bar{E},\ (j_{!*}(\mathcal{F}^K_\lambda \mathbb{V}[n]))[-n]),$$

where $j : \mathrm{Sh}_K \hookrightarrow \overline{\mathrm{Sh}_K}$ is the open embedding and $\mathcal{F}^K_\lambda \mathbb{V}$ is the λ-adic automorphic sheaf on Sh_K associated to an algebraic representation $\mathbb{V}$ of $G_{\bar{\mathbb{Q}}}$. In this introduction we may assume for simplicity that $\mathbb{V}$ is trivial and $\lambda = l$ is a rational prime, so that $\mathcal{F}^K_\lambda \mathbb{V} = \mathbb{Q}_l$. Let $p \neq l$ be a hyperspecial prime for K, in the sense that $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ a hyperspecial subgroup and $K^p \subset G(\mathbb{A}_f^p)$ a compact open subgroup. There are commuting actions of the prime-to p Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p)//K^p)$ and $\mathrm{Gal}(\bar{E}/E)$ on $\mathbf{IH}^*$. Fix $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p)//K^p)$, and let $\Phi = \Phi_{\mathfrak{p}}$ be the geometric Frobenius of a place $\mathfrak{p}$ of E lying above p. Let $j \in \mathbb{N}$.

Conjecture 1 (Kottwitz, cf. [40] §10)**.** *The action of* $\mathrm{Gal}(\bar{E}/E)$ *on* $\mathbf{IH}^*$ *is unramified at* $\mathfrak{p}$*, and under simplifying assumptions we have*

$$\mathrm{Tr}(f^{p,\infty} \times \Phi^j | \, \mathbf{IH}^*) = \sum_H \iota(G,H) ST^H(f^H). \tag{0.2.1}$$

In the summation H *runs through the elliptic endoscopic data of* G*. For each* H *the test function* $f^H \in C_c^\infty(H(\mathbb{A}), \mathbb{C})$ *is explicitly chosen in terms of its stable orbital integrals, and* ST^H *is the geometric side of the stable trace formula.*

In fact, Kottwitz also formulates the following conjecture:

Conjecture 2 (Kottwitz, cf. [40] §7)**.** *Let* $\mathbf{H}_c^*$ *be the compact support étale cohomology of* $\mathrm{Sh}_K \times \mathrm{Spec}\, \bar{E}$*. The action of* $\mathrm{Gal}(\bar{E}/E)$ *on* $\mathbf{H}_c^*$ *is unramified at* $\mathfrak{p}$*, and under simplifying assumptions we have*

$$\mathrm{Tr}(f^{p,\infty} \times \Phi^j | \, \mathbf{H}_c^*) = \sum_H \iota(G,H) ST_e^H(f^H). \tag{0.2.2}$$

Here H *and* f^H *are the same as in Conjecture 1, and* ST_e^H *is the elliptic part of* ST^H*.*

Compared to (0.2.1), both sides of (0.2.2) in Conjecture 2 are easier to compute, at least from a group theoretical point of view. In fact, Kottwitz ([40][41]) uses the Grothendieck-Lefschetz trace formula to compute the LHS of (0.2.2) for certain PEL Shimura varieties, by counting (virtual) abelian varieties with additional structures over finite fields. For these Shimura varieties Kottwitz proves that the LHS of (0.2.2) is equal to a group theoretical expression of the form

$$\sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0,\gamma,\delta) O_\gamma(f^{p,\infty}) TO_\delta(\phi) \operatorname{Tr}(\gamma_0,\mathbb{V}) \tag{0.2.3}$$

involving orbital integrals and twisted orbital integrals on G. He conjectures that the LHS of (0.2.2) is equal to (0.2.3) in general (under group theoretical simplifying assumptions). He then *stabilizes* (0.2.3) in [40], in the sense that he finds out the definitions of the test functions f^H and proves that (0.2.3) is equal to the RHS of (0.2.2). [1] Here the RHS of (0.2.2) has a relatively simple definition in terms of stable orbital integrals.

The motivation for formulating Conjecture 1 in addition to Conjecture 2 is mainly due to the fact that in general it is difficult to extract spectral information from ST^H_e. Thus it is natural to replace ST^H_e by ST^H, of which ST^H_e can be viewed as the leading term in a suitable expansion. On the LHS of (0.2.1), the motivation for introducing $\mathbf{IH}^*$ is from Arthur's work on L2-cohomology and Zucker's conjecture. Recall that Zucker's conjecture (proved by Looijenga and Saper-Stern) states that $\mathbf{IH}^*$ is isomorphic to the L2-cohomology of $\mathrm{Sh}_K(\mathbb{C})$ as Hecke modules. Arthur [1] shows that the trace of a Hecke operator on the L2-cohomology of an arbitrary locally symmetric space can be expressed in terms of his *invariant trace formula*, and an analogue of Conjecture 1 without the Frobenius follows from Arthur's stabilization of the invariant trace formula ([3][2][4]). We mention that Goresky-Kottwitz-MacPherson [18] prove the same formula as [1] for the trace of a Hecke operator on the intersection cohomology. Thus Conjecture 1 can be regarded as a refinement of the work of Arthur and Goresky-Kottwitz-MacPherson on the trace of a Hecke operator mentioned above, with extra information on the Galois action.

Conjecture 1 had been regarded as much more difficult than Conjecture 2, due to the lack of effective methods of computing $\mathbf{IH}^*$. This difficulty was overcome by the work of

[1] The definition of $f^{H,p,\infty}$ involves the Langlands-Shelstad transfer and the fundamental lemma, so any proof of the identities (0.2.1) or (0.2.2) prior to the establishment of those is conditional.

Morel [56][57] (also cf. [58]). Morel is able to prove Conjecture 1 for Siegel modular spaces ([60]) and $\mathrm{GU}(p,q)$-Shimura varieties ([59]).

0.3 The main result

The aim of this work is to prove Conjecture 1 for orthogonal Shimura varieties for almost all primes. These are Shimura varieties associated to the Shimura datum (G, X), where $G = \mathrm{SO}(V)$ is the special orthogonal group of a quadratic space V over $\mathbb{Q}$ of signature $(n, 2)$, and X is the set of oriented negative definite planes in $V_{\mathbb{R}}$. We assume $n \geq 5$. The reflex field is $\mathbb{Q}$ and the Shimura varieties are of dimension n. These Shimura varieties are non-compact and not of PEL type. They are typical examples of abelian type Shimura varieties of types B and D. More precisely, recall from [15] that for type B there is essentially a unique family of Shimura data, and for type D there are two. The orthogonal Shimura data represent the type B family and one of the two type D families. We remark that the Shimura data treated by Kottwitz and Morel are of types A and C.

Throughout this work we make the *distinction of odd and even cases*, according to the parity of n.

Theorem 1 (Corollary 6.17.4)**.** *Conjecture 1 is true for the orthogonal Shimura varieties, for almost all primes p and for divisible enough j.*

Remark 1. As mentioned in §0.2, in this work we allow a non-trivial automorphic sheaf as the coefficient system of the cohomology. In the even case a technical assumption on the sheaf is needed. cf. the statement of Corollary 6.17.4.

Remark 2. We have stated Theorem 1 for almost all primes p, rather than for all hyperspecial p. This is due to the lack of the proof, at least in the written literature, of the existence of good integral models at hyperspecial primes of the minimal and toroidal compactifications of the orthogonal Shimura varieties. This issue should be resolved by the ongoing work of Madapusi Pera.

Remark 3. When proving Theorem 1, we assume Conjecture 2 for the orthogonal Shimura varieties. In fact, Conjecture 2 for more general Shimura varieties is proved in a joint work with Kisin and Shin [32]. See the discussion in §0.4.2.

Remark 4. In stating Conjectures 1 and 2 in the forms above, we have made several simplifying technical assumptions, about the possibility of embedding ${}^L H$ into ${}^L G$ and about the shape of Z_G. The general forms of the conjectures are not difficult to formulate, by considering z-extensions and central data. cf. [32]. These simplifying assumptions are satisfied by the orthogonal Shimura data.

Remark 5. Kottwitz indicates ([40] Part II) how Conjecture 1 together with Arthur's multiplicity conjecture implies a full description of $\mathbf{IH}^*$ as a module over $\mathrm{Gal}(\bar{E}/E) \times \mathcal{H}(G(\mathbb{A}_f)//K)$ in terms of global Arthur parameters. See also [25]. Note that Arthur's multiplicity conjecture has recently been established in some cases, with substitutes of Arthur parameters. See for instance [5] and [28].

Remark 6. For $n \equiv 3, 5 \mod 8$, there is a unique quadratic space V over $\mathbb{Q}$ of signature $(n, 2)$ and split at all finite places. Taïbi's results [73] on Arthur's multiplicity conjecture apply to $G = \mathrm{SO}(V)$. In this case one expects to obtain a full description of $\mathbf{IH}^*$ from Theorem 1. We hope to pursue this in a future work.

0.4 Idea of the proof: reduction to the boundary comparison

We will use the following two inputs to reduce the proof of Theorem 1 to a boundary comparison.

1. Morel's formula for $\mathrm{Tr}(f^{p,\infty} \times \Phi^j | \mathbf{IH}^*)$. See Theorem 2 below.

2. The proof of Conjecture 2 in cases including the orthogonal Shimura varieties, from [32]. See identity (0.4.2) below.

0.4.1 Input (1)

We fix the choice of a minimal parabolic P_{12} of G and a Levi subgroup $M_{12} \subset P_{12}$. The group G has three standard parabolic subgroups $P_1, P_2, P_{12} = P_1 \cap P_2$. For P a standard parabolic subgroup, we write M_P for the corresponding standard Levi subgroup. We have

$$M_{P_1,\mathbb{R}} = \mathrm{GL}_2 \times \mathrm{SO}(n-2,0)$$

$$M_{P_2,\mathbb{R}} = \mathbb{G}_m \times \mathrm{SO}(n-1,1)$$

$$M_{P_{12},\mathbb{R}} = \mathbb{G}_m^2 \times \mathrm{SO}(n-2,0).$$

Morel's work ([56][57][59]) implies an expansion of $\mathrm{Tr}(f^{p,\infty} \times \Phi^j | \mathbf{IH}^*)$ of the following form.

Theorem 2 (Theorem 1.7.2)**.**

$$\mathrm{Tr}(f^{p,\infty} \times \Phi^j | \mathbf{IH}^*) = \mathrm{Tr}_G + \mathrm{Tr}_{M_{P_{12}}} + \mathrm{Tr}_{M_{P_1}} + \mathrm{Tr}_{M_{P_2}}. \tag{0.4.1}$$

For the precise definitions of the terms see §2.

In the above expansion (0.4.1),

$$\mathrm{Tr}_G = \mathrm{Tr}(f^{p,\infty} \times \Phi^j | \mathbf{H}^*_{\text{ét},c}(\mathrm{Sh}_{K,\bar{\mathbb{Q}}})),$$

and it is known to be given by Kottwitz's fixed point formula (0.2.3) and can be stabilized as in [40]. We will come back to this point in §0.4.2 below. For $P \in \{P_1, P_2, P_{12}\}$, the term Tr_{M_P} is more complicated in its structure. It is a mixture of the following:

- Kottwitz's fixed point formula (0.2.3) for a boundary stratum in $\overline{\mathrm{Sh}_K}$, which is roughly speaking a smaller Shimura variety itself.

- The (topological) fixed point formula of Goresky-Kottwitz-MacPherson [18], for the trace of a Hecke operator on the compact support cohomology of a locally symmetric space.

- *Kostant-Weyl traces.* By this we mean Weyl character formulas for certain algebraic sub-representations of $M_{\tilde{P}}$ of

$$\mathbf{H}^*(\mathrm{Lie}\, N_{\tilde{P}}, \mathbb{V}),$$

  where $\tilde{P}$ is a standard parabolic subgroup of G containing P, and $N_{\tilde{P}}$ is its unipotent radical. Here $\mathbb{V}$ is the algebraic representation of $G_{\bar{\mathbb{Q}}}$ used to define the automorphic λ-adic sheaf in §0.2. These sub-representations of $\mathbf{H}^*(\mathrm{Lie}\, N_{\tilde{P}}, \mathbb{V})$ are cut out by the weights of certain central cocharacters of $M_{\tilde{P}}$.

Roughly speaking, the terms Tr_{M_P} arise because Morel's work reduces computing $\mathrm{Tr}(\mathbf{IH}^*)$ to computing traces on the compact support cohomology of each stratum in the Baily-Borel compactification. These compact support cohomology groups are taken with non-trivial coefficient systems. The definition of these coefficient systems involve

- cohomology of certain arithmetic groups, which are computed by the fixed point formula of Goresky-Kottwitz-MacPherson

- (weight truncations of) cohomology of $\mathrm{Lie}\, N_{\tilde{P}}$, which give rise to the Kostant-Weyl traces.

Once the coefficient system is known, the compact support cohomology of a stratum in this coefficient system is computed by Kottwitz's fixed point formula.

Because of group theoretical differences, the terms in (0.4.1) are defined slightly differently than those in Morel's original formula in [59] [60]. For more details see §2.

0.4.2 Input (2)

In a collaborated work with Kisin and Shin [32], we stabilize the term Tr_G for quite general Hodge type and some abelian type Shimura varieties, including the orthogonal Shimura varieties considered in this work, building on Kisin's work [29] and [30]. Namely, we prove Conjecture 2 in those cases (by proving that the LHS of (0.2.2) is equal to the generalization of (0.2.3) and by stabilizing the latter in a way similar to Kottwitz's stabilization in [40]). Thus we have the identity

$$\mathrm{Tr}_G = \sum_H \iota(G,H) ST_e^H(f^H). \tag{0.4.2}$$

In view of (0.4.1) and (0.4.2), we see that in order to prove Theorem 1 we need only compare the boundary terms. Namely, it suffices to prove the following theorem.

Theorem 3 (Theorem 6.17.1)**.**

$$\mathrm{Tr}_{M_{P_1}} + \mathrm{Tr}_{M_{P_2}} + \mathrm{Tr}_{M_{P_{12}}} = \sum_H \iota(G,H)[ST^H(f^H) - ST_e^H(f^H)]. \tag{0.4.3}$$

Remark 0.4.3. The logical dependence of this work on [32] is only formal. In fact, only the very last statement Corollary 6.17.4 in this work depends on the main result of loc. cit.

0.5 Idea of the proof: the boundary comparison

We now present some ingredients in the proof of Theorem 3.

To calculate the RHS of (0.4.3), we use the formulas of Kottwitz in his unpublished notes, which are valid because of the specific form of f^H_∞. A comparison of these formulas with Arthur's stabilization has not appeared in the literature, and we hope to pursue this in future work. From the point of view taken in this work, we take Kottwitz's formulas, recalled in §6.3, as the definitions of $ST^H(f^H)$.

According to Kottwitz's formulas, we have an expansion of the form

$$ST^H(f^H) - ST^H_e(f^H) = \sum_{M' \neq H} ST^H_{M'}(f^H),$$

where M' runs through proper Levi subgroups of H, and each term $ST^H_{M'}(\cdot)$ has a relatively simple formula.

Roughly speaking, we classify the pairs (H, M') appearing in the above summation into four different classes, labeled by the standard Levi subgroups M of G and the symbol $\emptyset$. We write $(H, M') \sim M$, or $(H, M') \sim \emptyset$. In order to prove Theorem 3, we only need to show

$$\mathrm{Tr}_M = \sum_{(H,M') \sim M} ST^H_{M'}(f^H), \text{ for } M \in \{M_{P_1}, M_{P_2}, M_{P_{12}}\} \tag{0.5.1}$$

and

$$0 = \sum_{(H,M') \sim \emptyset} ST^H_{M'}(f^H). \tag{0.5.2}$$

In the proof of (0.5.1) and (0.5.2), key ingredients include the following:

Archimedean comparison.

We need to compare archimedean contributions to both sides of (0.5.1), and compute the archimedean contributions to the RHS of (0.5.2).

For the RHS of (0.5.1) (0.5.2), the archimedean contributions consist of values of (stable) discrete series characters on non-elliptic maximal tori. We will carry out explicit computation of these using formulas due to Harish Chandra [22] and Herb [23].

For the LHS of (0.5.1), the archimedean contributions are understood to be the Kostant-Weyl traces discussed below Theorem 2. The main tool we use to compute these is Kostant's Theorem [33].

A key ingredient in the proof of (0.5.1) is an identity between the archimedean contributions to both sides. We remark that as for now the desired archimedean comparison does not have a general conjectural form, and they are established case by case for Gsp ([60]), $\mathrm{GU}(p,q)$ ([59]), and for $\mathrm{SO}(V)$ in this work. It would be interesting to find a more conceptual framework predicting such a comparison.

Computation at p.

We need to compute the contributions from the place p to both sides of (0.5.1) and the RHS of (0.5.2). Roughly speaking, the boundary of $\overline{\mathrm{Sh}_K}$ is stratified into modular curves and zero-dimensional Shimura varieties, and the contributions from p to the LHS of (0.5.1) count mod p points on modular curves and zero dimensional Shimura varieties. On the RHS of (0.5.1), the contributions from p fall into two kinds. One "good kind" that resemble the contributions from p to the LHS, and one "bad kind" that does not have a clear geometric interpretation. We will need to prove, among other things, that the terms on the RHS of (0.5.1) having a bad factor at p should eventually cancel each other.

Comparing transfer factors.

Because we aim to prove non-trivial cancellations among terms on the RHS of (0.5.1) and aim to prove the vanishing of (0.5.2), signs are important. One important non-trivial source of signs come from comparing different normalizations of transfer factors at infinity. The necessity of this step is overlooked in [59] and [60]. For the orthogonal groups of interest in this work, it turns out that a non-trivial pattern of signs arises from such comparisons, which is essential in the proof of (0.5.1) and (0.5.2). The comparison in question is between the normalization $\Delta_{j,B}$ introduced in [40] §7, and the Whittaker normalization, for the groups of interest. We always explicitly fix G (and also its standard Levi subgroups) as a pure inner form of its quasi-split inner form, so the Whittaker normalization for the transfer factors between G and an endoscopic group can be defined, once an equivalence class of Whittaker data has been chosen for G^*. (Similarly for a standard Levi subgroup of G and its endoscopic groups). In the odd case, there is only one equivalence class of Whittaker data, since G^* is adjoint. When $d = \dim V$ is even and $d/2$ is odd, there is also a unique equivalence class of Whittaker data. The most difficult case is when d is divisible by 4, when there are two equivalence classes of Whittaker data for G^*, and we need to figure out how the two corresponding Whittaker normalizations behave when we pass to Levi subgroups. This we achieve in turn by comparing the Whittaker normalizations with $\Delta_{j,B}$, and comparing the explicit formula for $\Delta_{j,B}$ with the explicit formula given by Waldspurger [81]. In this direction we prove Theorem 4.7.15, which could be of independent interest.

0.6 Organization of the paper

In §1, we recall the global settings of orthogonal Shimura varieties and state Morel's formula (0.4.1) in this case in Theorem 1.7.2. Its proof is postponed to §2.

In §2, we define all the terms in Theorem 1.7.2 and prove it.

In §3, we first review some generalities on quasi-split quadratic spaces. These will be used when we define endoscopic groups.

§4 consists of the archimedean calculation in this work. There are two different parts of the calculation: the part on endoscopy and especially the comparison between different normalizations of the transfer factors, and the part on the archimedean comparison between the Kostant-Weyl traces and the stable discrete series characters. We carry out the first part in §4.3-§4.8, and the second part in §4.9-§4.11.

In §5, we first review the endoscopic data in the global and local non-archimedean cases in §5.1, and then in §5.2-§5.4 we carry out some Satake transform calculations at p that will be needed later in the stabilization.

In §6, we first review some standard facts on Langlands-Shelstad transfer in §6.1. We then recall the definition (according to Kottwitz) of $ST^H(f^H)$ in §6.2 and §6.3, and recall the definition of f^H in §6.4. After that we prove what amounts to (0.5.1) in §6.5-§6.14, and we prove what amounts to (0.5.2) in §6.15 and §6.16. We deduce the main result of this work in Corollary 6.17.4.

On first reading, the reader is advised to only focus on the odd case for simplicity.

0.7 Some conventions

- All reductive groups are assumed to be connected.

- For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer $\leq x$ and denote by $\lceil x \rceil$ the smallest integer $\geq x$.

- We denote by $]0,1[$ the interval $\{x \in \mathbb{R} | 0 < x < 1\}$.

- We use the letter a, as opposed to the usage of the letter j in many references, to denote the integer exponent of the Frobenius under consideration.

- Let (V_1, q_1) be a finite dimensional quadratic space over a field F of characteristic zero. Its determinant $\det q_1$ is defined as an element of $F^\times/F^{\times,2}$, given by the determinant of the Gram matrix of a basis of V_1. We define the *discriminant*, an element of $F^\times/F^{\times,2}$, as

$$\delta := \begin{cases} \det q_1, & \dim V \text{ odd.} \\ (-1)^{\dim V/2} \det q_1, & \dim V \text{ even.} \end{cases}$$

  The convention in the even case is simply to make any orthogonal direct sum of hyperbolic planes have discriminant $1 \in F^\times/F^{\times,2}$.

- Whenever a quadratic form is fixed from the context, we shall use the notation $[\cdot,\cdot]$ to denote its associated bilinear form.

- The letter m and its derived notations, such as $m_1, m^\pm$, are usually used to denote the absolute rank of a special orthogonal group.

- For $a \in \mathbb{N}$ and p a prime number, we denote by $\mathbb{Q}_{p^a}$ the degree a unramified extension of $\mathbb{Q}_p$, and we denote by $\mathbb{Z}_{p^a}$ the valuation ring of $\mathbb{Q}_{p^a}$. We denote by σ the p-Frobenius acting on $\mathbb{Q}_{p^a}$.

- If G_1 is a reductive group over $\mathbb{Q}$, we denote by $\ker^1(\mathbb{Q}, G_1)$ the kernel set

$$\ker(\mathbf{H}^1(\mathbb{Q}, G_1) \to \mathbf{H}^1(\mathbb{A}, G_1)).$$

- We will sometimes use the following abbreviations:

$$\Gamma = \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \quad \Gamma_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \quad \Gamma_{\mathbb{R}} = \Gamma_\infty = \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

- By the *standard type B root datum* of rank m we mean the root datum

$$(\mathbb{Z}^m = \mathrm{span}_{\mathbb{Z}} \{\epsilon_1, \cdots, \epsilon_m\}, R, \mathbb{Z}^m = \mathrm{span}_{\mathbb{Z}}(\epsilon_1^\vee, \cdots, \epsilon_m^\vee), R^\vee)$$

with the standard pairing $\mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ (under which the natural bases $\{\epsilon_i\}$ and $\{\epsilon_i^\vee\}$ are dual to each other), such that the roots are given by

$$R = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i | 1 \leq i, j \leq m, i \neq j\}$$

and the coroots are given by

$$R^\vee = \left\{\pm\epsilon_i^\vee \pm \epsilon_j^\vee, \pm 2\epsilon_i | 1 \leq i, j \leq m, i \neq j\right\}.$$

The *natural order* is the following choice of simple roots:

$$\epsilon_1 - \epsilon_2, \cdots, \epsilon_{m-1} - \epsilon_m, \epsilon_m.$$

Note that the Weyl group of this root datum is equal to $\{\pm 1\}^m \rtimes \mathfrak{S}_m$. Here $\{\pm 1\}^m$ acts on $\mathbb{Z}^m$ by the product action of the action of $\{\pm 1\}$ on $\mathbb{Z}$ given by multiplication. The symmetric group in m letters $\mathfrak{S}_m$ acts on $\mathbb{Z}^m$ by permuting the coordinates, and also on $\{\pm 1\}^m$ by permuting the coordinates.

We shall also call $(\mathbb{Z}^m, R, \mathbb{Z}^m, R^\vee)$ as above together with the natural order the *standard type B based root datum.*

- Similarly, by the *standard type D root datum* of rank m we mean the root datum

$$(\mathbb{Z}^m, R, \mathbb{Z}^m, R^\vee)$$

such that the roots are given by

$$R = \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i, j \leq m, i \neq j\}$$

and the coroots are given by

$$R^\vee = \left\{\pm\epsilon_i^\vee \pm \epsilon_j^\pm | 1 \leq i, j \leq m, i \neq j\right\}.$$

The *natural order* is the following choice of simple roots:

$$\epsilon_1 - \epsilon_2, \cdots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m.$$

Note that the Weyl group of this root datum is equal to $(\{\pm 1\}^m)' \rtimes \mathfrak{S}_m$, where $(\{\pm 1\}^m)'$ is the subgroup of $\{\pm 1\}^m$ consisting of elements with the property that an even number of the coordinates are equal to -1.

We shall also call $(\mathbb{Z}^m, R, \mathbb{Z}^m, R^\vee)$ as above together with the natural order the *standard type D based root datum.*

- In this work we use the language of abelian Galois cohomology as developed in [43]. We follow the notation there. We also use Kottwitz's language of the same theory (which is the original form of the theory) involving centers of Langlands dual groups, as in [38].

- When normalizing transfer factors, we use the classical normalization of local class field theory as opposed to Deligne's normalization. This means in particular that in the non-archimedean case the uniformizers correspond to the arithmetic Frobenius. cf. [35, §4.1, §4.2].

- When considering Whittaker data over $\mathbb{R}$, we always use the following additive character:

$$\psi_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}^{\times}$$

$$x \mapsto e^{2\pi i x}.$$

  When defining the epsilon factors over $\mathbb{R}$ we always use the above additive character $\psi_{\mathbb{R}}$ and the usual Lebesgue measure on $\mathbb{R}$, the latter being self-dual with respect to $\psi_{\mathbb{R}}$.

- When considering perverse sheaves, we exclusively consider the middle perversity.

1 Geometric setting

1.1 Groups

Fix V to be a quadratic space over $\mathbb{Q}$, of signature $(n, 2)$. We will denote the quadratic form by q and the associated bilinear form by $[\cdot, \cdot]$. Let $d = \dim V = n + 2$. Let $G = \mathrm{SO}(V)$. Suppose $n \geq 5$, so that among the maximal isotropic subspaces of $V_{\mathbb{R}}$, of dimension two, there is one defined over $\mathbb{Q}$. We fix a flag

$$0 \subset V_1 \subset V_2 \subset V_2^\perp \subset V_1^\perp \subset V, \tag{1.1.1}$$

where V_i is an i-dimensional totally isotropic $\mathbb{Q}$-subspace of V. Define

$$P_1 := \mathrm{Stab}_G(V_2) \subset G$$

$$P_2 := \mathrm{Stab}_G(V_1) \subset G$$

$$P_{12} := P_1 \cap P_2 \subset G.$$

Then P_{12} is a minimal $\mathbb{Q}$-parabolic subgroup of G, and P_1, P_2 are the standard $\mathbb{Q}$-parabolic subgroups of G with respect to P_{12}. The parabolic subgroups P_1, P_2 are maximal.

If S is a non-empty subset of $\{1, 2\}$, we write P_S for the one of P_1, P_2, P_{12}, corresponding to S.

We fix once and for all a splitting of the flag (1.1.1). Then we get Levi subgroups M_S of P_S, which are standard with respect to the minimal Levi subgroup M_{12}. We have

$$M_1 \cong \mathrm{GL}(V_2) \times \mathrm{SO}(V_2^\perp/V_2),$$

$$M_2 \cong \mathrm{GL}(V_1) \times \mathrm{SO}(V_1^\perp/V_1),$$

$$M_{12} \cong \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(V_2^\perp/V_2).$$

In the sequel we will only consider the standard parabolic subgroups and the standard Levi subgroups with respect to (P_{12}, M_{12}). If $P = P_S$ is a standard parabolic subgroup, we write $M_P := M_S$ for its standard Levi subgroup. We write $N_P = N_S$ for the unipotent radical of P.

We fix a basis e_1 of V_1 and a basis e_2 of V_2/V_1. Using the chosen splitting of (1.1.1), we can think of e_2 as a vector in V. Let $e_1' \in V/V_1^\perp$ and $e_2' \in V_1^\perp/V_2^\perp$ be determined by the conditions $[e_i, e_i'] = 1, i = 1, 2$. We can think of e_i' as vectors of V, as well. Under these choices we have identifications

$$\mathrm{GL}(V_i) \cong \mathrm{GL}_i,\ i = 1, 2.$$

Definition 1.1.1. We write $W_i := V_i^\perp/V_i$ for $i = 1, 2$.

Definition 1.1.2.

$$M^{\mathrm{GL}} = \begin{cases} \mathrm{GL}(V_2), & M = M_1 \\ \mathrm{GL}(V_1), & M = M_2 \\ \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1), & M = M_{12} \end{cases}$$

$$M^{\mathrm{SO}} = \begin{cases} \mathrm{SO}(W_2), & M = M_1 \\ \mathrm{SO}(W_1), & M = M_2 \\ \mathrm{SO}(W_2), & M = M_{12} \end{cases}$$

Hence we have $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$ for $M \in \{M_1, M_2, M_{12}\}$.

1.2 Shimura datum

We recall the orthogonal Shimura datum.

Consider the space X of oriented, negative definite, two dimensional subspaces $\mathbf{h} \subset V_{\mathbb{R}}$. The real manifold X can be identified with the set

$$X' := \{z \in V_{\mathbb{C}} - \{0\} \,|q(z) = 0, [z, \bar{z}] < 0\} /\mathbb{C}^\times$$

in the following way. Given $\mathbf{h} \in X$, take an oriented $\mathbb{R}$-basis x, y of $\mathbf{h}$, such that

$$[x, y] = 0, \quad q(x) = q(y).$$

We send $\mathbf{h}$ to $z := x + iy \in \mathcal{D}'$. Conversely, starting with $z \in X'$, we write $z = x + iy$ with $x, y \in V_{\mathbb{R}}$. The conditions satisfied by z imply that

$$q(x) = q(y) < 0, \quad [x, y] = 0.$$

We send z to $\mathbf{h} := \mathbb{R}x + \mathbb{R}y$, with orientation given by (x, y).

Under the identification $X \cong X'$, the space X is equipped with the structure of a complex manifold, and in fact a Hermitian symmetric space.

The space $X \cong X'$ has two connected components, and the natural action of $G(\mathbb{R}) = \mathrm{SO}(V)(\mathbb{R})$ on X induces the non-trivial action of $\pi_0(G(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ on $\pi_0(X)$.

Given a point $\mathbf{h} \in \mathcal{X}$, choose x, y to be an oriented basis of $\mathbf{h}$ satisfying $[x, y] = 0$ and $q(x) = q(y) = -1$. For $a + bi \in \mathbb{C}^\times$, $a, b \in \mathbb{R}$, define $\bar{\alpha}_{\mathbf{h}}(a + bi) \in G(\mathbb{R})$ as the action $(a + bxy)(\cdot)(a + bxy)^{-1}$ on $V_{\mathbb{R}} = \mathbf{h} + \mathbf{h}^\perp$. Then $\bar{\alpha}_{\mathbf{h}}(re^{i\theta})$ acts on $V_{\mathbb{R}}$ by rotation on $\mathbf{h}$ of angle -2θ and identity on $\mathbf{h}^\perp$.

The map $\bar{\alpha}_{\mathbf{h}} : \mathbb{C}^\times \to G(\mathbb{R})$ comes from a homomorphism between $\mathbb{R}$-algebraic groups

$$\alpha_{\mathbf{h}} : \mathbb{S} \to G_{\mathbb{R}}.$$

Also the association $\mathbf{h} \mapsto \alpha_{\mathbf{h}}$ is $G(\mathbb{R})$-equivariant and identifies X as a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$. Under this construction the pair (G, X) is a Shimura datum. We denote the Shimura datum (G, X) by $\mathbf{O}(V)$. It has reflex field $\mathbb{Q}$, and the associated Shimura varieties have dimension $n = d - 2$.

The Hodge cocharacter $\mu : \mathbb{G}_m \to G$ (up to $G(\bar{\mathbb{Q}})$-conjugacy) of $\mathbf{O}(V)$ is given as follows.

Choose a decomposition of V into an orthogonal direct sum of $m = \lfloor d/2 \rfloor$ hyperbolic planes (plus a line in the odd case) over $\bar{\mathbb{Q}}$. For the i-th hyperbolic plane, choose a basis x_i, y_i such that

$$q(x_i) = q(y_i) = 0, \quad [x_i, y_i] = 1.$$

Then we obtain a maximal torus

$$\mathbb{G}_m^m \hookrightarrow G_{\bar{\mathbb{Q}}},$$

such that the i-th $\mathbb{G}_m$ acts with weight 1 on x_i and weight -1 on y_i and acts trivially on all the other hyperbolic planes (and the line in the odd case). Let $\epsilon_1^\vee, \cdots, \epsilon_m^\vee$ be the standard basis of $X_*(\mathbb{G}_m^m)$. Then μ is conjugate to $\epsilon_1^\vee$. Note that in $G_{\bar{\mathbb{Q}}}$, all $\epsilon_1^\vee, \cdots, \epsilon_m^\vee$ are conjugate to each other, and $\epsilon_1^\vee$ is conjugate to $-\epsilon_1^\vee$. In particular μ is conjugate to μ^{-1}. Moreover, it is possible to find a representative $\mu : \mathbb{G}_m \to G$ of the Hodge cocharacter defined over $\mathbb{Q}$. In fact, we may assume that the aforementioned decomposition of $V_{\bar{\mathbb{Q}}}$ into hyperbolic planes (and a line in the odd case) involves the hyperbolic plane span $\{e_1, e_1'\}$ as the first member. Then the cocharacter $\epsilon_1^\vee$ is defined over $\mathbb{Q}$.

The Shimura datum $\mathbf{O}(V)$ is of abelian type. It is the quotient by $\mathbb{G}_m$ of the Gspin Shimura datum, which is of Hodge type. For more details see for example [55].

1.3 Boundary components

We describe the pure Shimura data associated to the boundary components of $\mathbf{O}(V)$ in the sense of Pink ([62]). It suffices to discuss the maximal standard parabolic subgroups P_1 and P_2.

For P_1, consider its Levi subgroup $M_1 = \mathrm{GL}(V_2) \times \mathrm{SO}(V_2^\perp / V_2)$. The factor $\mathrm{GL}(V_2)$ is the image of Pink's canonical subgroup of P_1. The resulting pure Shimura datum for $\mathrm{GL}(V_2) \cong \mathrm{GL}_2$ is the Siegel Shimura datum $\mathbf{H}_1$.

For P_2, consider its Levi subgroup $M_1 = \mathrm{GL}(V_1) \times \mathrm{SO}(V_1^\perp / V_1)$. The factor $\mathrm{GL}(V_1)$

is the image of Pink's canonical subgroup of P_2. The resulting pure Shimura datum for $\mathrm{GL}(V_1) \cong \mathbb{G}_m$ is the Siegel Shimura datum $\mathbf{H}_0$.

For a proof of these facts see [24, Proposition 2.4.5].

1.4 Shimura varieties

For any neat compact open subgroup K of $G(\mathbb{A}_f)$, we have the canonical model $\mathrm{Sh}_K(G, X) = \mathrm{Sh}_K$ for $\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{C})\backslash X \times G(\mathbb{A}_f)/K$, which is a smooth quasi-projective variety of dimension $n = d-2$ over $\mathbb{Q}$. Since the Shimura data $\mathbf{O}(V)$ is of abelian type, the construction of the canonical models is given by Deligne ([13] and [15]).

Let K_1, K_2 be neat compact open subgroups of $G(\mathbb{A}_f)$ and let $g \in G(\mathbb{A}_f)$. Suppose $K_1 \subset gK_2g^{-1}$. Denote by $[\cdot g]_{K_1,K_2}$ the map

$$\mathrm{Sh}_{K_1}(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_1 \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_2$$

$$[(x, k)] \mapsto [(x, kg)].$$

This map descends to a finite étale $\mathbb{Q}$-variety morphism $\mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$, which we also denote by $[\cdot g]_{K_1,K_2}$. When the context is clear we will simply write $[\cdot g]$. We call $[\cdot g]$ a *Hecke operator*.

1.5 The Baily-Borel compactification

For any neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the Shimura variety Sh_K has a canonical Baily-Borel compactification

$$j : \mathrm{Sh}_K \hookrightarrow \overline{\mathrm{Sh}_K},$$

where $\overline{\mathrm{Sh}_K}$ is a normal projective variety over $\mathbb{Q}$ and j is a dense open embedding defined over $\mathbb{Q}$. The Hecke operators $[\cdot g]$ have canonical extensions to finite morphisms between the

Baily-Borel compactifications, defined over $\mathbb{Q}$. Denote the extension by $\overline{[\cdot g]}$. These facts are proved in Pink's thesis [62, 12.3]

1.6 Automorphic λ-adic sheaves

Let $\mathbb{V}$ be a finite dimensional vector space over a number field L with an algebraic representation of G. Let λ be a place of L. Then it is well known (cf. [63, §5.1]) that for any neat compact open subgroup $K_1 \subset G(\mathbb{A}_f)$, there is an associated λ-adic sheaf on Sh_{K_1}, denoted by $\mathcal{F}_\lambda^{K_1}\mathbb{V}$. Moreover, for a Hecke operator $[\cdot g]_{K_1,K_2} : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2}$, there is a canonical isomorphism

$$\mathcal{F}_{[\cdot g]_{K_1,K_2}} : \mathcal{F}_\lambda^{K_1}\mathbb{V} \xrightarrow{\sim} ([\cdot g]_{K_1,K_2})^* \mathcal{F}_\lambda^{K_2}\mathbb{V}. \tag{1.6.1}$$

We denote the system $(\mathcal{F}_\lambda^K\mathbb{V}, \mathcal{F}_{[\cdot g]})$ simply by $\mathcal{F}_\lambda\mathbb{V}$, and call it *the system of automorphic λ-adic sheaves associated to $\mathbb{V}$ on* $\mathrm{Sh}(\mathbf{O}(V))$.

Remark 1.6.1. For a general Shimura datum (G, X), the construction of the automorphic sheaves either depends on the assumption that Z_G has equal $\mathbb{Q}$-split rank and $\mathbb{R}$-split rank or depends on extra hypotheses on the central character of the representation $\mathbb{V}$. In our case this is not an issue, because the group $\mathrm{SO}(V)$ is semi-simple.

Definition 1.6.2. Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. Define the intersection complex

$$\mathrm{IC}_\lambda^K\,\mathbb{V} := (j_{!*}(\mathcal{F}_\lambda^K\mathbb{V}[n]))[-n].$$

Here j is the open embedding $\mathrm{Sh}_K \hookrightarrow \overline{\mathrm{Sh}_K}$, and recall that $n = \dim \mathrm{Sh}_K$.

In the following discussion we sometimes omit the subscript λ from the notation. Now we recall the intersection complex analogues of the canonical isomorphisms (1.6.1).

Consider a Hecke operator

$$[\cdot g] : \mathrm{Sh}_{K_1} \to \mathrm{Sh}_{K_2},$$

with its canonical extension

$$\overline{[\cdot g]} : \overline{\mathrm{Sh}_{K_1}} \to \overline{\mathrm{Sh}_{K_2}}.$$

To ease notation we denote $g := [\cdot g]$ and $\bar{g} := \overline{[\cdot g]}$. Denote by j_i the open embedding $\mathrm{Sh}_{K_i} \to \overline{\mathrm{Sh}_{K_i}}$, $i = 1, 2$.

Since both g and $\bar{g}$ are proper, we have base change morphisms

$$\bar{g}^* Rj_{2,*} \to Rj_{1,*} g^*$$

$$\bar{g}^* Rj_{2,!} \to Rj_{1,!} g^*$$

filling into the commutative diagram

$$\begin{tikzcd}
\bar{g}^* Rj_{2,!} \arrow[r] \arrow[d] & Rj_{1,!} g^* \arrow[d] \\
\bar{g}^* Rj_{2,*} \arrow[r] & Rj_{1,*} g^*
\end{tikzcd}$$

where the vertical maps are induced by the natural maps $Rj_{i,!} \to Rj_{i,*}$, $i = 1, 2$. Since $\bar{g}$ is finite (cf. §1.5), it is exact with respect to the perverse t-structure and commutes with ${}^p\mathcal{H}^0$ (the zeroth perverse cohomology). Therefore the above commutative diagram induces a natural transformation

(1.6.2) $$\bar{g}^* j_{2,!*} \to j_{1,!*} g^*.$$

Plugging in $\mathcal{F}^{K_2}\mathbb{V}[n]$ we get

$$\bar{g}^* j_{2,!*}(\mathcal{F}^{K_2}\mathbb{V}[n]) \to j_{1,!*}g^*(\mathcal{F}^{K_2}\mathbb{V}[n]),$$

which we further compose as follows:

$$\bar{g}^* j_{2,!*}(\mathcal{F}^{K_2}\mathbb{V}[n]) \to j_{1,!*}g^*(\mathcal{F}^{K_2}\mathbb{V}[n]) \xrightarrow{j_{1,!*}(\mathcal{F}^{-1}_{[\cdot g]})} j_{1,!*}(\mathcal{F}^{K_1}\mathbb{V}[n]).$$

Shifting by $[-n]$ we obtain a map

$$g^* \operatorname{IC}^{K_2}\mathbb{V} \to \operatorname{IC}^{K_1}\mathbb{V}. \tag{1.6.3}$$

Similarly, using the base change morphisms

$$Rj_{1,!}g^! \to \bar{g}^! Rj_{2,!}$$

$$Rj_{1,*}g^! \to \bar{g}^! Rj_{2,*},$$

we obtain the following analogue of (1.6.2):

$$j_{1,!*}g^! \to \bar{g}^! j_{2,!*}. \tag{1.6.4}$$

Note that $g^! = g^*$ because g is finite étale (cf. §1.4). Again, plugging $\mathcal{F}^{K_2}\mathbb{V}[n]$ into (1.6.4), precomposing with $j_{1,!*}(\mathcal{F}_{[\cdot g]})$ and shifting by $[-n]$ we obtain a map

$$\operatorname{IC}^{K_1}\mathbb{V} \to \bar{g}^! \operatorname{IC}^{K_2}\mathbb{V}. \tag{1.6.5}$$

We summarize the above construction of the two maps (1.6.3) (1.6.5) in the following statement: For Hecke operators $[\cdot g_1]_{K',K_1}$ and $[\cdot g_2]_{K',K_2}$, we have a canonical cohomological correspondence

$$\mathscr{H}_{[\cdot g_1]_{K',K_1},\ [\cdot g_2]_{K',K_2}} :\ \overline{[\cdot g_1]}^*_{K',K_1} \mathrm{IC}^{K_1}\,\mathbb{V} \to \overline{[\cdot g_2]}^!_{K',K_2} \mathrm{IC}^{K_2}\,\mathbb{V}. \tag{1.6.6}$$

1.7 Intersection cohomology and Morel's formula

Let K be a neat compact open subgroup of $G(\mathbb{A}_f)$. The λ-adic intersection cohomology of $\overline{\mathrm{Sh}_K}$ with coefficients in $\mathbb{V}$ is by definition:

$$\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda := \mathbf{H}^*_{\text{ét}}(\overline{\mathrm{Sh}_K} \times_{\mathbb{Q}} \bar{\mathbb{Q}},\ \mathrm{IC}^K_\lambda\,\mathbb{V}).$$

By abuse of notation we also denote by $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda$ the formal alternating direct sum

$$\bigoplus_i (-1)^i\,\mathbf{IH}^i(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda.$$

We have an action of

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}_{L_\lambda}(G(\mathbb{A}_f)//K)$$

on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda$, where $\mathcal{H}_{L_\lambda}(G(\mathbb{A}_f)//K)$ is the Hecke algebra consisting of the L_λ-valued functions in $C_c^\infty(G(\mathbb{A}_f), L)$ that are bi-invariant under K, and it acts via the cohomological correspondences (1.6.6). After choosing an isomorphism $\bar{L}_\lambda \cong \mathbb{C}$, we get an action of $\mathcal{H}(G(\mathbb{A}_f)//K) := \mathcal{H}_{\mathbb{C}}(G(\mathbb{A}_f)//K)$ on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda \otimes_{L_\lambda} \mathbb{C}$.

More precisely, to normalize the action of $\mathcal{H}_{L_\lambda}(G(\mathbb{A}_f)//K)$ on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda$ we need to fix the choice of a Haar measure dg on $G(\mathbb{A}_f)$ that takes rational values on compact open subgroups. We take the convention that for any $g \in G(\mathbb{A}_f)$, the function $1_{gKg^{-1}}/\operatorname{vol}_{dg}(K)$ acts by the endomorphism on $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda$ induced by the cohomological correspondence

(1.6.6) with

$$g_1 = g, \quad g_2 = 1, \quad K_1 = K_2 = K, \quad K' = gKg^{-1} \cap K.$$

We follow the convention of [59, Remark 1.6.7] on how a cohomological correspondence such as (1.6.6) induces an endomorphism on cohomology. This is different from the convention in [18].

In the following, for $f \in \mathcal{H}(G(\mathbb{A}_f)//K)$, we will talk about the action of fdg on

$$\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda \otimes_{L_\lambda} \mathbb{C},$$

rather than the action of f, in order to emphasize the choice of dg. This notation is compatible with the fact that the action of $f(Cdg)$ is equal to the action of $(Cf)dg$, for any $C \in \mathbb{C}^\times$.

We also denote by $\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})_\lambda$ (the formal alternating direct sum of) the compact support étale cohomology

$$\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})_\lambda := \mathbf{H}^*_{\text{ét},c}(\mathrm{Sh}_K \times_{\mathbb{Q}} \bar{\mathbb{Q}},\ \mathcal{F}^K_\lambda \mathbb{V}).$$

Similarly we have an action of $\mathcal{H}(G(\mathbb{A}_f)//K)$ on $\mathbf{H}^*_c(\mathrm{Sh}_K, \mathbb{V})_\lambda \otimes_{L_\lambda} \mathbb{C}$ after the choice of a Haar measure.

Definition 1.7.1. Let K be an open compact subgroup of $G(\mathbb{A}_f)$ and let $p \in \mathbb{Z}$ be a prime. We say that p is a *hyperspecial prime* for K, if $K = K_pK^p$ with K_p a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}^p_f)$.

Theorem 1.7.2 (Morel's formula)**.** *Fix $\mathbb{V}$ and λ as before. Let $\ell \in \mathbb{Z}$ be the prime number beneath λ. Fix a neat compact open subgroup K of $G(\mathbb{A}_f)$. Let Σ_0 be the finite set consisting of $2, \ell$, and the prime numbers that are not hyperspecial for K. There exists a finite set $\Sigma = \Sigma(\mathbf{O}(V), \mathbb{V}, \lambda, K)$ of prime numbers, containing Σ_0, such that for all prime numbers $p \notin \Sigma$*

the following is true. Let $\mathrm{Frob}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *be a representative of the geometric Frobenius at* p*. Let* $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p), K^p)$ *be arbitrary. Let* $a \in \mathbb{N}$ *be a sufficiently divisible (depending on* $\mathbb{V}, \lambda, K, p,$ *and* $f^{p,\infty}$*) but otherwise arbitrary number. Let* dg^p *be a Haar measure on* $G(\mathbb{A}_f^p)$ *and let* dg_p *be the Haar measure on* $G(\mathbb{Q}_p)$ *giving volume* 1 *to* K_p*. Write* $dg := dg^p dg_p$ *for the product Haar measure on* $G(\mathbb{A}_f)$*. The actions of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on* $\mathbf{IH}^*(\overline{\mathrm{Sh}_K}, \mathbb{V})_\lambda$ *and on* $\mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})_\lambda$ *are both unramified at* p*, and we have*

(1.7.1)

$$\mathrm{Tr}(\mathrm{Frob}_p^a \times f^{p,\infty} 1_{K_p} dg, \mathbf{IH}^*(\overline{\mathrm{Sh}_K},\ \mathbb{V})_\lambda \otimes \mathbb{C}) = \mathrm{Tr}(\mathrm{Frob}_p^a \times f^{p,\infty} 1_{K_p} dg, \mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})_\lambda \otimes \mathbb{C}) + \\ + \mathrm{Tr}_{M_1}(f^{p,\infty} dg^p, a) + \mathrm{Tr}_{M_2}(f^{p,\infty} dg^p, a) + \mathrm{Tr}_{M_{12}}(f^{p,\infty} dg^p, a),$$

where the definitions of the terms

$$\mathrm{Tr}_{M_1}(f^{p,\infty} dg^p, a), \quad \mathrm{Tr}_{M_2}(f^{p,\infty} dg^p, a), \quad \mathrm{Tr}_{M_{12}}(f^{p,\infty} dg^p, a)$$

are given in §2.

We will prove Theorem 1.7.2 in §2.7. Here we remark that the above theorem is essentially proved by Morel [59], but some modifications to her proof are needed in our situation.

Remark 1.7.3. Once we have a good theory of integral models for the Baily-Borel as well as the toroidal compactifications of Sh_K at all odd hyperspecial primes, the above Theorem 1.7.2 could be improved to be valid for $\Sigma = \Sigma_0$. Note that such a theory for Hodge type Shimura varieties has already been established by Madapusi Pera [54]. We have been informed that the extension of the results in loc. cit. to the case of abelian type including the orthogonal Shimura varieties is expected to be worked out by Madapusi Pera in the near future.

2 Morel's formula

In this section we define the terms in Theorem 1.7.2 and prove Theorem 1.7.2. We will define the terms Tr_M in Definition 2.6.1. The definition also involves *Kostant-Weyl terms* L_M to be defined below and notions from Kottwitz's fixed point formula to be recalled below.

2.1 A decomposition of the Levi subgroups

- For

$$M_{12} = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(V_2^{\perp}/V_2) = \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SO}(V_2^{\perp}/V_2),$$

  we call $\mathrm{GL}(V_1)$ the first $\mathbb{G}_m$ and $\mathrm{GL}(V_2/V_1)$ the second $\mathbb{G}_m$. Define

$$M_{12,h} := \mathrm{GL}(V_1)$$

  and

$$M_{12,l} := \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(V_2^{\perp}/V_2).$$

- For

$$M_1 = \mathrm{GL}(V_2) \times \mathrm{SO}(V_2^{\perp}/V_2),$$

  define

$$M_{1,h} := \mathrm{GL}(V_2)$$

  and

$$M_{1,l} := \mathrm{SO}(V_2^{\perp}/V_2).$$

  As above we refer to $\mathrm{GL}(V_1)$ and $\mathrm{GL}(V_2/V_1)$ as the first and second $\mathbb{G}_m$ respectively.

- For

$$M_2 = \mathrm{GL}(V_1) \times \mathrm{SO}(V_1^{\perp}/V_1),$$

define

$$M_{2,h} := \mathrm{GL}(V_1)$$

and

$$M_{2,l} := \mathrm{SO}(V_1^{\perp}/V_1).$$

If $M \in \{M_1, M_2, M_{12}\}$ is a standard Levi subgroup of G, then with the above definition we have $M = M_h \times M_l$.

Remark 2.1.1. The decomposition $M = M_h \times M_l$ is different from the decomposition $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$ in Definition 1.1.2 when $M = M_{12}$.

2.2 Definition of some truncations

Definition 2.2.1. We define the following objects.

- Let ϖ_1 be the standard central cocharacter of $\mathrm{GL}(V_2) \cong \mathrm{GL}_2$.
- Let ϖ_2 be the square of the identity cocharacter of $\mathrm{GL}(V_1) \cong \mathbb{G}_m$.
- Define $t_1 := 1 - (d-2) = 3 - d$ and $t_2 := 0 - (d-2) = 2 - d$.

Remark 2.2.2. For $i = 1, 2$ we know that ϖ_i is a cocharacter of A_{M_i}, the maximal $\mathbb{Q}$-split torus of the center of M_i. Moreover, ϖ_1 (resp. ϖ_2) is the weight cocharacter of the Siegel Shimura datum $(M_{1,h}, \mathbf{H}_1) = (\mathrm{GL}_2, \mathbf{H}_1)$ (resp. $(M_{2,h}, \mathbf{H}_0) = (\mathbb{G}_m, \mathbf{H}_0)$), and t_1 (resp. t_2) is equal to the (complex) dimension of the Shimura varieties associated to $(M_{1,h}, \mathbf{H}_1)$ (resp. $(M_{2,h}, \mathbf{H}_0)$) minus that of the orthogonal Shimura variety $\mathrm{Sh}_K(G, X)$.

Definition 2.2.3. In this definition all the algebraic groups involved are understood to have been base changed to $\mathbb{C}$. We also base change $\mathbb{V}$ to $\mathbb{C}$ and still use $\mathbb{V}$ to denote it. For a standard parabolic subgroup P of G, define a virtual M_P-representation $R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V})_{>t_P}$ as follows (where t_P is just a symbol):

- For $P = P_{12}$, it is the sub-object of the virtual M_{12}-representation

$$R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V}) := \bigoplus_i (-1)^i\, \mathbf{H}^i(\mathrm{Lie}\, N_P, \mathbb{V}),$$

on which ϖ_1 has weights $> t_1$ and ϖ_2 has weights $> t_2$.

- For $P = P_1$ (resp. P_2), it is the sub-object of the virtual M_1-representation (resp. M_2-representation) $R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V})$, on which ϖ_1 (resp. ϖ_2) has weights $> t_1$ (resp. $> t_2$).

When $P = P_{12}$, we also write $R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V})_{>t_{12}}$ for $R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V})_{>t_P}$.

2.3 Definition of the Kostant-Weyl term $L_{M_{12}}$

Let $M = M_{12}$.

Definition 2.3.1. Let $\mathcal{P}(M)$ be the set of pairs (P, g), where P is a standard parabolic of G (i.e. $P \in \{P_1, P_2, P_{12}\}$), and $g \in G(\mathbb{Q})$, satisfying

- the composition $M_h \xrightarrow{\mathrm{Int}(g)} P \to M_P$ maps M_h into $M_{P,h}$, and the map differs from the inclusion $M_h \hookrightarrow M_{P,h}$ by an inner automorphism of M_P.

- the composition $M_l \xrightarrow{\mathrm{Int}(g)} P \to M_P$ embeds M_l into $M_{P,l}$ as a Levi subgroup.

For $(P, g), (P', g') \in \mathcal{P}(M)$, we define them to be *equivalent* if and only if $P = P'$ and there exist $h_1 \in M_P(\mathbb{Q})$ and $h_2 \in M(\mathbb{Q})$ such that $g' = h_1 g h_2$. Here M_P is the standard Levi subgroup of P. Denote this equivalence relation on $\mathcal{P}(M)$ by $\sim$. For P a standard parabolic subgroup, denote by $N(P)$ the cardinality of the subset of $\mathcal{P}(M)/\sim$ consisting of elements whose first coordinate is P.

Definition 2.3.2. Define an element $n_{12} \in G(\mathbb{Q})$ in the odd case (resp. $n_{12} \in \mathrm{O}(V,q)(\mathbb{Q})$ in the even case) as follows:

$$n_{12}(e_1) = e_1,\ n_{12}(e_1') = e_1'$$

$$n_{12}(e_2) = e_2',\ n_{12}(e_2') = e_2$$

$$n_{12}|_{V_2^{\perp}/V_2} = -1.$$

Lemma 2.3.3. *We have*

$$\mathcal{P}(M)/\sim \ = \{(P_2, g)\}/\sim \bigsqcup \{(P_{12}, g)\}/\sim .$$

The quotient $\{(P_2, g)\}/\sim$ *is a singleton and can be represented by* $(P_2, 1)$*. The quotient* $\{(P_{12}, g)\}/\sim$ *has two elements, represented by* $(P_{12}, 1)$ *and* (P_{12}, n_{12})*. The map* $\mathrm{Int}(n_{12})$ *sends* M_h *into itself as the identity, and sends* $M_l = \mathrm{GL}(V_2/V_1) \times \mathrm{SO}(V_2^{\perp}/V_2)$ *into itself, not as the identity, but being the inversion on the* $\mathrm{GL}(V_2/V_1) = \mathbb{G}_m$ *factor and the identity on the* $\mathrm{SO}(V_2^{\perp}/V_2)$ *factor. Moreover,* $N(P_{12}) = 2$ *and* $N(P_2) = 1$*.*

Proof. This follows directly from the definitions. □

Remark 2.3.4. In the even case, $\mathrm{Int}\, n_{12}$ on G is a non-inner automorphism.

Definition 2.3.5. For $\gamma \in M(\mathbb{R})$, define the *Kostant-Weyl term*

$$L_M(\gamma) := 2 \sum_{(P,g)\in\mathcal{P}(M)/\sim} N(P)^{-1}(-1)^{\dim A_{gMg^{-1}}/A_{M_P}}$$

$$(n^{M_P}_{gMg^{-1}})^{-1} \left| D^{M_P}_{gMg^{-1}}(g\gamma g^{-1}) \right|^{1/2} \delta^{1/2}_{P(\mathbb{R})}(g\gamma g^{-1})\, \mathrm{Tr}(g\gamma g^{-1}, R\Gamma(\mathrm{Lie}\, N_P, \mathbb{V})_{>t_P}),$$

where

- $(n^{M_P}_{gMg^{-1}})^{-1} := |\mathrm{Nor}_{M_P}(gMg^{-1})(\mathbb{Q})/(gMg^{-1})(\mathbb{Q})|^{-1}$.

- $\mathrm{Nor}_{M_P}(gMg^{-1})$ is the normalizer of gMg^{-1} in M_P.

- A_{M_P} (resp. $A_{gMg^{-1}}$) is the maximal $\mathbb{Q}$-split torus in the center of M_P (resp. gMg^{-1}). Note that the base change to $\mathbb{R}$ of A_{M_P} (resp. $A_{gMg^{-1}}$) is the maximal $\mathbb{R}$-split torus in the center of $M_{P,\mathbb{R}}$ (resp. $(gMg^{-1})_{\mathbb{R}}$).[2]

- $D^{M_P}_{gMg^{-1}}(g\gamma g^{-1}) = \det\left(1 - \mathrm{ad}(g\gamma g^{-1}),\ \ \mathrm{Lie}(M_P)_{\mathbb{R}}/\,\mathrm{Lie}(gMg^{-1})_{\mathbb{R}}\right) \in \mathbb{R}$.

- the absolute value $\left|D^{M_P}_{gMg^{-1}}\right|$ is the real absolute value.

- $\delta_{P(\mathbb{R})}(g\gamma g^{-1}) = \left|\det\left(\mathrm{ad}(g\gamma g^{-1}), \mathrm{Lie}(N_P)_{\mathbb{R}}\right)\right|$. (Real absolute value.)

Remark 2.3.6. The factor 2 in the definition of $L_M(\gamma)$ is denoted by $m_{\mathbf{Q}}$ in [60, §1.2]. It accounts for the fact that for the zero dimensional Siegel Shimura datum $(\mathbb{G}_m, \mathbf{H}_0)$, the set $\mathbf{H}_0$ is a two-fold cover of a $\mathbb{G}_m(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{R}}.$$

Its role is discussed in the proof of Theorem 1.7.2 in §2.7.

2.4 Definition of the Kostant-Weyl terms L_{M_1} and L_{M_2}

Definition 2.4.1. For $\gamma \in M_1(\mathbb{R})$, define

$$L_{M_1}(\gamma) := \delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\,\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}(N_1), \mathbb{V})_{>t_1}),$$

where $\delta_{P_1(\mathbb{R})}$ is defined in Definition 2.3.5.

[2]In general one should use the dimensions of the maximal $\mathbb{R}$-split tori in the centers for the definition of L_M.

Definition 2.4.2. For $\gamma \in M_2(\mathbb{R})$, define

$$L_{M_2}(\gamma) := 2\delta_{P_2(\mathbb{R})}(\gamma)^{1/2}\operatorname{Tr}(\gamma, R\Gamma(\mathrm{Lie}(N_2), \mathbb{V})_{>t_2}),$$

where $\delta_{P_2(\mathbb{R})}$ is defined in Definition 2.3.5.

Remark 2.4.3. The factor 2 in the above definition is the same as in Remark 2.3.6.

2.5 Definition of some terms in Kottwitz's fixed point formula

Let M_h be the reductive group GL_i over $\mathbb{Q}$, where $i = 1$ or 2. We equip M_h with the Siegel Shimura datum $\mathbf{H}_g$ with $g = i - 1$. We recall some group theoretical definitions that appear in Kottwitz's fixed point formula for the Shimura varieties associated to $(M_h, \mathbf{H}_g)$. The main reference is [40]. See also [59, §1.6]. We note that these notions have been extended to more general situations (e.g. when M_h^{der} is not necessarily simply connected) in [32]. Here we only need very degenerated forms of the notions since GL_1 and GL_2 are very simple.

Throughout §2.5 we fix a prime p, an integer $a \in \mathbb{N}$, and a compact open subgroup $K^p_{M_h}$ of $M_h(\mathbb{A}^p_f)$.

Definition 2.5.1. Define a cocharacter μ of M_h as follows. When $M_h = \mathbb{G}_m$, define μ to be the identity cocharacter. When $M_h = \mathrm{GL}_2$, define μ to be the cocharacter

$$z \mapsto \begin{pmatrix} z & \\ & 1 \end{pmatrix}.$$

Remark 2.5.2. μ is a representative of the Hodge cocharacter for the Shimura datum $\mathbf{H}_g$.

Definition 2.5.3 (cf. [40] or [32])**.** A *Kottwitz triple* (of level p^a) in M_h (for the Shimura datum $\mathbf{H}_g$) is a triple $(\gamma_0, \gamma, \delta)$, with $\gamma_0 \in M_h(\mathbb{Q})$, $\gamma \in M_h(\mathbb{A}^p_f)$, $\delta \in M_h(\mathbb{Q}_{p^a})$, satisfying the following conditions:

- γ_0 is semi-simple and $\mathbb{R}$-elliptic.
- γ is conjugate to γ_0 in $M_h(\mathbb{A}_f^p)$.
- The element $\mathrm{N}\,\delta := \delta\sigma(\delta)\cdots\sigma^{a-1}(\delta) \in M_h(\mathbb{Q}_{p^a})$ is conjugate to γ_0 in $M_h(\mathbb{Q}_{p^a})$.
- the image of $[\delta] \in B(M_{h,\mathbb{Q}_p})$ in $\pi_1(M_h)_{\Gamma_p}$ under the Kottwitz map $B(M_{h,\mathbb{Q}_p}) \to \pi_1(M_h)_{\Gamma_p}$ is equal to that of $-\mu$.

Two Kottwitz triples $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are said to be *equivalent*, if γ_0 is conjugate to γ_0' and δ is σ-conjugate to δ' inside $M_h(\mathbb{Q}_{p^a})$.

Remark 2.5.4. In the above definition we have taken advantage of the fact that inside M_h stable conjugacy is the same as conjugacy. As a consequence the coordinate γ plays no role at all. For more general definitions see the references given above.

Remark 2.5.5. Note that Γ_p acts trivially on $\pi_1(M_h)$, and there is an isomorphism $\pi_1(M_h) \cong \mathbb{Z}$, such that the image of μ is $1 \in \mathbb{Z}$.

Let $(\gamma_0, \gamma, \delta)$ be a Kottwitz triple. Let $I_0 = (M_h)_{\gamma_0}$ be the (connected) centralizer of γ_0 in M_h. Define $\mathfrak{K}(I_0/\mathbb{Q})$ to be the finite abelian group consisting of those elements of $\pi_0([Z(\widehat{I_0})/Z(\widehat{M_h})]^\Gamma)$ whose images in $\mathbf{H}^1(\Gamma, Z(\widehat{M_h}))$ are locally trivial. (cf. [38].) In [40] Kottwitz defines an invariant $\alpha(\gamma_0, \gamma, \delta)$ in the group $\mathfrak{K}(I_0/\mathbb{Q})^D$ which is the dual group of $\mathfrak{K}(I_0/\mathbb{Q})$.

Lemma 2.5.6. *For $M_h = \mathrm{GL}_1$ or GL_2, we always have $\mathfrak{K}(I_0/\mathbb{Q}) = 1$.*

Proof. If $I_0 = M_h$ then obviously $\mathfrak{K}(I_0/\mathbb{Q}) = 1$. Thus we may assume that $M_h = \mathrm{GL}_2$ and γ_0 in non-central. Then I_0 is a maximal torus T of GL_2 defined over $\mathbb{Q}$. In this case $Z(\widehat{I_0}) = \widehat{I_0} = \widehat{T}$. Since the Galois action on $Z(\widehat{\mathrm{GL}_2})$ is trivial, by Chebotarev density theorem the only locally trivial element of $\mathbf{H}^1(\Gamma, Z(\widehat{\mathrm{GL}_2}))$ is the trivial element. In view of the exact sequence

$$\pi_0(Z(\widehat{\mathrm{GL}_2})^\Gamma) \to \pi_0(\widehat{T}^\Gamma) \to \pi_0([\widehat{T}/Z(\widehat{\mathrm{GL}_2})]^\Gamma) \to \mathbf{H}^1(\Gamma, Z(\widehat{\mathrm{GL}_2})),$$

we see that it suffices to show that

$$\widehat{T}^{\Gamma} \subset Z(\widehat{\mathrm{GL}_2}).$$

Since $T_{\mathbb{R}}$ is an elliptic maximal torus of $\mathrm{GL}_{2,\mathbb{R}}$, there exists an identification $\widehat{T} \cong \mathbb{C}^\times \times \mathbb{C}^\times$ such that the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\widehat{T}$ by switching the two coordinates. It follows that $\widehat{T}^{\Gamma_\infty} \subset Z(\widehat{\mathrm{GL}_2})$, and a fortiori $\widehat{T}^{\Gamma} \subset Z(\widehat{\mathrm{GL}_2})$. $\square$

Let $(\gamma_0, \gamma, \delta)$ be a Kottwitz triple. By Lemma 2.5.6, its invariant $\alpha(\gamma_0, \gamma, \delta)$ automatically vanishes. Hence as in [40, §3], there is an inner form I of I_0 over $\mathbb{Q}$, such that

- $I_{\mathbb{R}}$ is anisotropic mod center.
- For $v \notin \{\infty, p\}$, $I_{\mathbb{Q}_v}$ is the trivial inner form of I_0.
- $I_{\mathbb{Q}_p}$ is isomorphic to the twisted-centralizer $(M_h)^{\sigma}_{\delta}$ of δ in M_h (denoted by $I(p)$ in loc. cit.) as inner forms of I_0.

See loc. cit. for more general and more precise statements.

Fix Haar measures on $I(\mathbb{Q}_p), I(\mathbb{A}_f^p), I(\mathbb{R})$ such that the product Haar measure on $I(\mathbb{A})$ is the Tamagawa measure. Fix a Haar measure on $M_h(\mathbb{Q}_{p^a})$ such that $M_h(\mathbb{Z}_{p^a})$ has volume 1. Fix Haar measures on $M_h(\mathbb{R}), M_h(\mathbb{A}_f^p)$ arbitrarily.

Definition 2.5.7. Define $c(\gamma_0, \gamma, \delta) = c_1(\gamma_0, \gamma, \delta) c_2(\gamma_0, \gamma, \delta)$, where

$$c_1(\gamma_0, \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q}) \backslash I(\mathbb{A}_f)) = \tau(I) \,\mathrm{vol}(A_{M_h}(\mathbb{R})^0 \backslash I(\mathbb{R}))^{-1}.$$

$$c_2(\gamma_0, \gamma, \delta) = \left| \ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M_h)) \right|.$$

Definition 2.5.8. For $f^p \in \mathcal{H}(M_h(\mathbb{A}_f^p)//K^p_{M_h})$, the orbital integral $O_\gamma(f^p)$ is defined using the fixed Haar measure on $M_h(\mathbb{A}_f^p)$ and the Haar measure on $(M_h)_\gamma(\mathbb{A}_f^p)$ transferred from

$I(\mathbb{A}_f^p)$. For

$$f_p \in \mathcal{H}(M_h(\mathbb{Q}_{p^a})//M_h(\mathbb{Z}_{p^a})),$$

the twisted orbital integral $TO_\delta(f_p)$ is defined using the fixed Haar measure on $M_h(\mathbb{Q}_{p^a})$ and the Haar measure on $(M_h)^\sigma_\delta(\mathbb{Q}_p)$ transferred from $I(\mathbb{Q}_p)$. For more details see [40, §3]

Definition 2.5.9. Define $\phi_a^{M_h} \in \mathcal{H}(M_h(\mathbb{Q}_{p^a})//M_h(\mathbb{Z}_{p^a}))$ to be the characteristic function of

$$M_h(\mathbb{Z}_{p^a})\mu(\varpi)^{-1}M_h(\mathbb{Z}_{p^a})$$

inside $M_h(\mathbb{Q}_{p^a})$, where ϖ is any uniformizer of $\mathbb{Q}_{p^a}$ (e.g. $\varpi = p$).

2.6 Definition of Tr_M

Definition 2.6.1. Let $P \in \{P_1, P_2, P_{12}\}$ be a standard parabolic subgroup of G and let $M = M_P$. Let dg^p be a Haar measure on $G(\mathbb{A}_f^p)$. Define

$$\mathrm{Tr}_M(f^{p,\infty}dg^p, a) := \sum_{\gamma_L} \iota^{M_l}(\gamma_L)^{-1}\chi((M_l)^0_{\gamma_L}) \sum_{\substack{(\gamma_0,\gamma,\delta) \\ \gamma_0 \sim_{\mathbb{R}} \gamma_L}} c(\gamma_0,\gamma,\delta)\delta^{1/2}_{P(\mathbb{Q}_p)}(\gamma_L\gamma_0)\cdot$$

$$\cdot O_{\gamma_L\gamma}(f_M^{p,\infty})O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)})TO_\delta(\phi_a^{M_h})L_M(\gamma_L\gamma_0),$$

where γ_L runs through the semi-simple conjugacy classes in $M_l(\mathbb{Q})$ that are $\mathbb{R}$-elliptic, and $(\gamma_0, \gamma_l, \delta)$ runs through the equivalence classes of Kottwitz triples in M_h as in Definition 2.5.3, (whose invariants are automatically trivial by Lemma 2.5.6), and satisfying the extra condition $\gamma_0 \sim_{\mathbb{R}} \gamma_L$ with the following meaning:

$$\gamma_0 \in M_h(\mathbb{R})^0 \Longleftrightarrow \gamma_L \in M_l(\mathbb{R})^0.$$

Moreover,

- $\iota^{M_l}(\gamma_L) := \left|M_{l,\gamma_L}(\mathbb{Q})/(M_l)^0_{\gamma_L}(\mathbb{Q})\right|$,

- $\delta_{P(\mathbb{Q}_p)}(\gamma_L\gamma_0) := \left|\det\left(\mathrm{ad}(\gamma_L\gamma_0), \mathrm{Lie}\, N_P\right)\right|_{\mathbb{Q}_p}$,

- $\chi((M_l)^0_{\gamma_L})$ is the Euler characteristic of the reductive group $(M_l)^0_{\gamma_L}$ over $\mathbb{Q}$ as defined in [18, §7.10],

- $f^{p,\infty}_M$ is the constant term of $f^{p,\infty}$ as defined in [18, §7.13]. This function depends on auxiliary choices, but its orbital integrals are well defined once all the relevant Haar measures are fixed.

- $M_l(\mathbb{Z}_p)$ is the hyperspecial subgroup of $M_l(\mathbb{Q}_p)$ given by

$$[K_p \cap (M_l(\mathbb{Q}_p)N_P(\mathbb{Q}_p))]/(K_p \cap N_P(\mathbb{Q}_p)).$$

  Equivalently, $M_l(\mathbb{Z}_p)$ as a subgroup of $M_l(\mathbb{Q}_p)$ is the image of $K_p \cap P(\mathbb{Q}_p)$ under the projection

$$P \to P/N_P \cong M = M_h \times M_l \to M_l.$$

The terms $\chi((M_l)^0_{\gamma_L})$, $O_{\gamma_L\gamma}(f^{p,\infty}_M)$, $O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)})$ all depend on extra choices of Haar measures. We choose arbitrary Haar measures on $M_l(\mathbb{A}^p_f)$, and on $(M_l)_{\gamma_L}(\mathbb{A}^p_f)$ and $(M_l)_{\gamma_L}(\mathbb{Q}_p)$ for each γ_L.

- Define the Haar measure on $M(\mathbb{A}^p_f) = M_h(\mathbb{A}^p_f) \times M_l(\mathbb{A}^p_f)$ by the Haar measure on $M_l(\mathbb{A}^p_f)$ chosen above and the Haar measure on $M_h(\mathbb{A}^p_f)$ chosen in §2.5.

- Use the Haar measure on $M(\mathbb{A}^p_f)$ and dg^p to define the constant term $f^{p,\infty}_M$.

- Use the Haar measure on $M(\mathbb{A}^p_f)$ and the Haar measure on $(M_l)_{\gamma_L}(\mathbb{A}^p_f)$ chosen above, and the Haar measure on $(M_h)_\gamma(\mathbb{A}^p_f)$ chosen in loc. cit. to define the orbital integral $O_{\gamma_L\gamma}$.

- Use the Haar measure on $M_l(\mathbb{Q}_p)$ giving volume 1 to $M_l(\mathbb{Z}_p)$, and the Haar measure on $(M_l)_{\gamma_L}(\mathbb{Q}_p)$ chosen above, to define $O_{\gamma_L}(1_{M_l(\mathbb{Z}_p)})$.
- Use the Haar measures on $(M_l)_{\gamma_L}(\mathbb{A}_f^p)$ and $(M_l)_{\gamma_L}(\mathbb{Q}^p)$ chosen above to define the product measure on $(M_l)^0_{\gamma_L}(\mathbb{A}_f)$, and use the latter to define $\chi((M_l)^0_{\gamma_L})$. (cf. [18, §7.10]).

Remark 2.6.2. When $P = P_1$ or P_{12}, every element in $M_l(\mathbb{Q})$ is automatically semi-simple $\mathbb{R}$-elliptic, because $M_l(\mathbb{R}) \cong \mathrm{SO}(n-2,0)(\mathbb{R})$ is compact. When $P = P_2$ and n is even, $(M_l)_\mathbb{R}$ does not have elliptic maximal tori (cf. §4.1), so there are no $\mathbb{R}$-elliptic elements of $M_l(\mathbb{Q})$. In this case $\mathrm{Tr}_M(f^{p,\infty}, a) = 0$.

Remark 2.6.3. When $P = P_1$ the condition $\gamma_0 \sim_\mathbb{R} \gamma_L$ is automatic. In fact, in this case we have $M_h(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$ and $M_l(\mathbb{R}) = \mathrm{SO}(n-2,0)(\mathbb{R})$, both connected. When $P = P_{12}$ or P_2, we have $\pi_0(M_h(\mathbb{R})) = \{\pm 1\}$ and $\pi_0(M_l(\mathbb{R})) = \{\pm 1\}$.

Remark 2.6.4. The factor $\iota^{M_l}(\gamma_l)$ is omitted in [60] but recorded in [59]. It occurs when deducing Theorem 1.6.6 of loc. cit. from [18].

2.7 Proof of Theorem 1.7.2

2.7.1 Background discussion

Our formula is almost the same as [60, Théorème 1.2.1], and it essentially follows from Morel's computation in [59, Chapter 1], which takes an axiomatic approach, building on her earlier work [56][57] and the work of Pink [63].

The arguments in [59, Chapter 1] also take as inputs Deligne's conjecture, proved at different levels of generality by Pink [64], Fujiwara [16] and Varshavsky [76], and the fixed point formula of Goresky-Kottwitz-MacPherson [18] and the fixed point formula of Kottwitz [41]. In our case, the fixed point formula of Kottwitz is to be applied to the non-open boundary pure Shimura data, namely the Siegel data $\mathbf{H}_0$ and $\mathbf{H}_1$. For $\mathbf{H}_0$, everything can be directly verified, but finding the precise statement (Proposition 2.8.1) that we need is more subtle. See the explanations about issue **(2)** below. For $\mathbf{H}_1$, it is of PEL type and hence the fixed point formula of Kottwitz is valid. The reason that in this current proof we do not need the analogue of the fixed point formula of Kottwitz for the boundary datum $\mathbf{O}(V)$ of $\mathbf{O}(V)$ itself (which is of abelian type and not of PEL type, cf. §1.2), is because the term corresponding to that datum enters the right hand side of (1.7.1) only abstractly as the contribution $\mathbf{H}_c^*(\mathrm{Sh}_K, \mathbb{V})_\lambda$. However, such an analogous formula is proved in the recent work [32] (building on [29] and [30]), and is eventually needed in Theorem 6.17.3 and Corollary 6.17.4.

There are three important differences between our case and the axiomatic approach of [59].

(1)

For $M = M_{12}$, our definition of L_M is slightly different from [60]. The definition of the set $\mathcal{P}(M)/\sim$ in [60, §1.2] allows $g \in G(\mathbb{Q})$ such that the map $\mathrm{Int}(g) : M_h \to M_{P,h}$[3] is not the identity (even up to an inner automorphism of the target) a priori.

(2)

For $P = P_2$, the boundary datum corresponding to P does not satisfy the axioms on p. 2 of [59]. Recall that the axioms state that the Levi quotient M of P should be a direct product of two reductive groups G_P and L_P, such that among other things, $G_P(\mathbb{R})$ acts transitively on the boundary Shimura datum and $L_P(\mathbb{R})$ acts trivially. In our case this is not true. We would like to take G_P to be $\mathrm{GL}(V_1) = \mathbb{G}_m \subset M$, and take L_P to be $\mathrm{SO}(V_1^\perp/V_1)$ with $L_{P,\mathbb{R}} \cong \mathrm{SO}(d-3,1)$. Now the boundary (pure) Shimura datum $\mathbf{H}_0$ for P is the two-pointed

[3] Morel uses the letter $\mathbf{Q}$ for our P.

set, with the action of $M(\mathbb{R})$ on it given by the map

$$M(\mathbb{R}) \to \pi_0(M(\mathbb{R})) \to \pi_0(G(\mathbb{R})) = \{\pm 1\}$$

and the non-trivial action of $\{\pm 1\}$ on the two-pointed set. We have

$$\pi_0(M(\mathbb{R})) = \pi_0(\mathbb{G}_m(\mathbb{R})) \times \pi_0(\mathrm{SO}(d-3,1)(\mathbb{R})) = \{\pm 1\} \times \{\pm 1\},$$

and the map $\pi_0(M(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ is the multiplication map

$$\{\pm 1\} \times \{\pm 1\} \to \{\pm 1\}.$$

Thus $L_P(\mathbb{R})$ does not act trivially on the boundary Shimura datum in question.

(3) Our Shimura datum $\mathbf{O}(V)$ is only of abelian type, and the construction of the integral model of the Baily-Borel compactification is more subtle.

To resolve **(1)**, note that in the case considered in [60], the subgroups M_h and $M_{P,h}$ of G are equal and isomorphic to a certain Gsp_{2i}. The only non-trivial outer automorphism of Gsp_{2i} is represented by the automorphism sending g to $c(g)^{-1}g$, where $c(g)$ is the symplectic similitude. This outer automorphism could not possibly be induced on M_h by $\mathrm{Int}(g)$ for any $g \in G(\mathbb{Q})$. Thus in [60], every element of $\mathcal{P}(M)$ according to Morel's a priori weaker definition indeed satisfies our Definition 2.3.1. In our case, there does exist $g \in G(\mathbb{Q})$ satisfying the definition of [60] that induces a non-trivial outer automorphism on $M_h \cong \mathbb{G}_m$.[4] We have ruled out this possibility in Definition 2.3.1. Correspondingly, we have defined $N(P,g)$ as in Definition 2.3.1, as opposed to $\left|\left[\mathrm{Nor}'_G(M)/\left(\mathrm{Nor}'_G(M) \cap g^{-1}M_P g\right)\right](\mathbb{Q})\right|^{-1}$ as in [60].

To resolve **(2)**, we need to systematically modify the discussion in [59, Chapter 1] for

[4]For example, consider $g \in G(\mathbb{Q})$ that switches e_1 and e_1', and switches e_2 and e_2', and acts as identity on $V_2^\perp/V_2$. Then $\mathrm{Int}\, g$ on M_l is the same as $\mathrm{Int}\, n_{12}$, but on $M_h = \mathrm{GL}(V_1) = \mathbb{G}_m$ it is the non-trivial automorphism.

zero dimensional boundary data. Before that, let us identify the geometric issue underlying **(2)**. Recall that in general, the boundary strata of the Baily-Borel compactification are mostly naturally isomorphic to finite quotients of Shimura varieties at certain natural levels. Sometimes they are "accidentally" isomorphic to Shimura varieties at some other levels. When this does not happen, we say that they are *geunine quotients*. The axioms of loc. cit. imply that there are no genuine quotients, namely all boundary strata are Shimura varieties. Then the fixed point formula obtained loc. cit. is a mixture of two formulas: Kottwitz's fixed point formula on these boundary Shimura varieties associated to the Hermitian part of the Levi subgroup, and the fixed point formula of Goresky-Kottwitz-MacPherson on the locally symmetric spaces associated to the linear part of the Levi subgroup. In our case, the zero dimensional boundary strata corresponding to P_2 are genuine quotients. We need to replace the Shimura variety of the Hermitian part by a *finite quotient* of it, and meanwhile replace the locally symmetric space of the linear part by a *finite covering* of it.

As for **(3)**, in [59, §1.3] Morel provides two approaches to the construction of the integral model of the Baily-Borel compactification, for which Pink's formula (cf. [59, Theorem 1.2.3] and [59, p.8 item (6)]) holds, among other things. The first approach, [59, Proposition 1.3.1], cites Lan's work [46] to construct the integral model away from a controlled finite set of bad primes, which is valid in the PEL type case. The second approach, [59, Proposition 1.3.4], works in much more general situations but only constructs the integral model away from an uncontrolled finite set of primes. It depends crucially on the generic base change theorem of Deligne [14, Th. finitude Théorème 1.9] in order to verify Pink's formula. Although Lan's work has been generalized by Madapusi Pera [54] to the Hodge type case, unfortunately our Shimura datum $\mathbf{O}(V)$ is only of abelian type (cf. §1.2), and the relevant treatment has not appeared in the literature. [5] As a result we could only use the second approach to construct

[5]Note that in this first approach, good integral models of toroidal compactifications are also implicitly needed in order to verify Pink's formula. Thus the "crude" construction in the abelian type case in [47] of the integral model of the Baily-Borel compactification alone is not sufficient.

the integral model away from an uncontrolled set of primes. Thus we would like to apply the abelian type case of [59, Proposition 1.3.4] to construct the integral model. Note that when stating conditions (1)-(7) on p. 8 of [59], Morel adopts notations that already assume that no genuine quotients happen on the boundary. However this is only a matter of notation, because the results from [63] and [82] used in the proof of [59, Proposition 1.3.4] do not need such an assumption. In the following we will talk about these conditions (1)-(7) with the understanding that their formulation have already been modified in the obvious way to suit the general possibility of genuine quotients. Thus [59, Proposition 1.3.4] is still valid in our case, and we could use it to construct the integral models of the Baily-Borel compactification as well as all of its strata over $\operatorname{Spec}\mathbb{Z}[1/\Sigma]$, for a large enough finite set of primes $\Sigma \supset \Sigma_0$, such that these integral models satisfy the conditions (1)-(7) on p. 8 of [59]. However, in the actual proof we proceed in a slightly different way and only construct these models over a finite étale Galois extension R of $\operatorname{Spec}\mathbb{Z}[1/\Sigma]$, with the reward of gaining more information about them.

2.7.2

Proof of Theorem 1.7.2. We now start to carry out the aforementioned modifications to [59, Chapter 1].

In the axioms on p. 2 of [59], we make the following modifications in case the boundary (pure) Shimura datum $\mathcal{X}_P$ corresponding to the admissible parabolic P is zero-dimensional. We weaken the axiom that $L_P(\mathbb{R})$ should act trivially on $\mathcal{X}_P$ to only requiring that $L_P(\mathbb{R})^0$ should act trivially. Let $L_P(\mathbb{Q})^+ := L_P(\mathbb{Q}) \cap L_P(\mathbb{R})^0$. For simplicity, we make the following assumptions:

- $|\mathcal{X}| = [L_P(\mathbb{Q}) : L_P(\mathbb{Q})^+] = 2$.
- $L_P(\mathbb{Q})/L_P(\mathbb{Q})^+$ acts non-trivially on $\mathcal{X}$.

- For any $\mathbb{Q}$-Levi subgroup L of L_P, we have $[L(\mathbb{Q}) : L(\mathbb{Q})^+] = 2$, where $L(\mathbb{Q})^+ := L(\mathbb{Q}) \cap L(\mathbb{R})^0$.

In particular, $L(\mathbb{Q})^+ = L(\mathbb{Q}) \cap L_P(\mathbb{Q})^+$ and $L(\mathbb{Q}) - L(\mathbb{Q})^+ = L(\mathbb{Q}) - L_P(\mathbb{Q})^+$.

In the following, if P is an admissible (standard) parabolic subgroup of G corresponding to zero dimensional boundary data, we need to always modify the definition of H_L by replacing $L(\mathbb{Q})$ with $L(\mathbb{Q})^+$, i.e. we define

$$H_L := gKg^{-1} \cap L_P(\mathbb{Q})^+ N_P(\mathbb{A}_f).$$

As on p. 2 of [59], the boundary strata in the Baily-Borel compactification corresponding to P are of the form

$$M^{K_Q/K_N}(G_P, \mathcal{X}_P)/H_P. \tag{2.7.1}$$

The action of H_P on $M^{K_Q/K_N}(G_P, \mathcal{X}_P)$ (Morel's notation for $\mathrm{Sh}_{K_Q/K_N}(G_P, \mathcal{X}_P)$) factors through the finite quotient $H_P/H_L K_Q$, now with the new definition of H_L. (We still define $H_P := gKg^{-1} \cap P(\mathbb{Q}) Q_P(\mathbb{A}_f)$.)

For boundary data of positive dimension, we keep the axioms and all the definitions unchanged.

With this new definition of H_L, the results of [59, §1.2] are still true. Here we still define $H := K_M \cap L(\mathbb{Q}) G(\mathbb{A}_f) = K_M \cap M(\mathbb{Q}) G(\mathbb{A}_f)$, not using $L(\mathbb{Q})^+ G(\mathbb{A}_f)$.

We now come to construct the integral models of the Baily-Borel compactification $\overline{\mathrm{Sh}_K}$ and its strata. As alluded to above in §2.7.1, our goal is to construct a finite set $\Sigma \supset \Sigma_0$ of primes, a finite étale Galois ring extension R of $\mathbb{Z}[1/\Sigma]$, and an integral model $(\overline{\mathrm{Sh}_K})_R$ (together with its strata) of $\overline{\mathrm{Sh}_K}$ over R, in the sense that $(\overline{\mathrm{Sh}_K})_R \otimes_R R_\eta$ is isomorphic to $\mathrm{Sh}_K \otimes_{\mathbb{Q}} R_\eta$ as R_η-schemes, where $R_\eta = R \otimes_{\mathbb{Z}} \mathbb{Q}$. For p a prime number not in Σ we denote

$R_{(p)} := R \otimes \mathbb{Z}_{(p)}$. Also denote $R_\eta := R \otimes_{\mathbb{Z}} \mathbb{Q}$. We require that $(\overline{\mathrm{Sh}_K})_R$ and its strata satisfy (the modified versions of) conditions (1)-(7) on p. 8 of [59], and that they satisfy the following conditions **(a)** and **(b)**:

(a) For all $p \notin \Sigma$, the base change to $R_{(p)}$ of any stratum in $(\overline{\mathrm{Sh}_K})_R$ is isomorphic to the base change to $R_{(p)}$ of the expected finite quotient of the hyperspecial canonical integral model of abelian type (cf. [29].) We also require that for all zero dimensional boundary strata, the finite set of bad primes as in Proposition 2.8.1 is contained in Σ.

(b) For all $p \notin \Sigma$, the model $(\overline{\mathrm{Sh}_K})_R \otimes R_{(p)}$ is the same as that constructed in [47, Proposition 2.4] (with a certain choice of an auxiliary Hodge type Shimura datum).

We construct $(\overline{\mathrm{Sh}_K})_R$ and its strata in Lemma 2.7.3 below. Now we admit this. Let a_0 be the degree of R over $\mathbb{Z}[1/\Sigma]$.

We continue to modify [59, Chapter 1]. Let p be an arbitrary prime not in Σ. Following Morel, we use notations $M^K(G, \mathcal{X})$ etc. to denote the reduction modulo p of the integral models over R constructed above. Here we view all of these reductions as varieties over $\mathbb{F}_{p^{a_0}}$.

With the new definition of H_L, the results of [59, §1.4, §1.5] all carry over. (In [59, Proposition 1.4.5], we replace $L_{n_r}(\mathbb{Q})$ with $L_{n_r}(\mathbb{Q})^+$ when defining H_L, in case n_r corresponds to zero dimensional data.) Note that in these sections Morel always uses notations that explicitly record the fact that the boundary strata are quotients of Shimura varieties. However, the comment on the top of p. 5 of [59], that $M^K(G, \mathcal{X})/H$ is equal to $M^{H/H_L}(G, \mathcal{X})$ and that H/H_L is a subgroup of $G(\mathbb{A}_f)$,[6] is no longer true in our case.

Also note that the change of the definition of H_L causes changes to some of the definitions before [59, Corollary 1.4.6] on p. 12 of [59]. The definition of X_L need not be modified. The definition of Γ_L is changed after the modification, involving the new definition of H_L. The new Γ_L is a subgroup of the old Γ_L. Thus the definitions of $R\Gamma(\Gamma_L, W)$ and $R\Gamma_c(\Gamma_L, W)$ loc.

[6] This comment says precisely that the axioms of [59] imply that none of the boundary strata are genuine quotients.

cit. are both changed, involving the new definition of $\Gamma_L\backslash X_L$. We call this new version of $\Gamma_L\backslash X_L$ *shallower locally symmetric spaces.*

We modify [59, Proposition 1.7.2] as follows. In the definitions of L, L_i, M, M_i, we modify the definitions of

$$R\Gamma(K_L, V), R\Gamma(H_{i,L}, V), R\Gamma_c(K_L, V), R\Gamma_c(H_{i,L}, V)$$

all using the corresponding shallower locally symmetric spaces (and also with the new definition $H_i \cap L^+(\mathbb{Q})$ for $H_{i,L}$). Here for $R\Gamma(K_L, V)$ and $R\Gamma_c(K_L, V)$, the shallower locally symmetric space in question is

$$L(\mathbb{Q})^+\backslash X_L \times L(\mathbb{A}_f)/K_L,$$

where $X_L = L(\mathbb{R})/K_{L,\infty}A_L(\mathbb{R})^0$ as on p. 12 of [59]. In the following we write $R\Gamma^{\mathrm{shall}}(K_L, V)$, $R\Gamma_c^{\mathrm{shall}}(K_L, V)$ etc. to emphasize that we are using shallower locally symmetric spaces. In the conclusions (1) (2) of [59, Proposition 1.7.2], the Shimura varieties M^{K_i} and M^K need to be replaced respectively by the finite quotients $M^{m_iK_Mm_i^{-1}\cap G(\mathbb{A}_f)}/H_i$ and $M^{K_M\cap G(\mathbb{A}_f)}/H$, where $H = K_M \cap L(\mathbb{Q})G(\mathbb{A}_f)$. (We are just undoing the application of the comment on p. 5 of [59] mentioned above.) Similar changes need to be made to [59, Lemma 1.7.3]. We still let $(m_i)_{i\in I}$ be a system of representatives for the double quotient $L(\mathbb{Q})G(\mathbb{A}_f)\backslash M(\mathbb{A}_f)/K'_M$ such that $cmK_M = cK_M$. Namely, we do not replace $L(\mathbb{Q})$ by $L^+(\mathbb{Q})$ here.

We can then modify the arguments on p. 25 of [59] by applying the modified [59, Proposition 1.7.2]. Of course modifications are only needed when the boundary datum corresponding to n_r is zero dimensional. Suppose this is the case. The definition of v_h needs to be changed. Again we simply undo the application of the comment on p. 5 of [59] mentioned before. Thus we replace the symbol $\mathcal{F}^{K_{M,h}/K_{L,h}}$ in Line 2 on p. 25 of [59] with $\mathcal{F}^{H_h/H_{L,h}}$. Also we change $R\Gamma_c(K_{L,h}, \cdot)$ to $R\Gamma_c^{\mathrm{shall}}(K_{L,h}, \cdot)$, defined using the shallower locally symmetric space

$$L_P(\mathbb{Q})^+ \backslash X_{L_P} \times L_P(\mathbb{A}_f)/K_{L,h}.$$

To calculate $\mathrm{Tr}(v_h)$, we need a modified version of Kottwitz's fixed point formula for zero-dimensional Shimura varieties ([59, Remark 1.6.5]), now for finite quotients of them, and we need a modified version of the fixed point formula of Goresky-Kottwitz-MacPherson for locally symmetric spaces ([59, Theorem 1.6.6]), now for shallower locally symmetric spaces.

In the next two subsections §2.8 §2.9 of this work, we formulate and prove these two analogues under simplifying assumptions that are satisfied by the boundary datum associated to P_2. Note that the factor 2 in Definition 2.3.5 and Definition 2.4.2 in this work corresponds to the factor 2 in Proposition 2.9.1 in this work, which is an analogue of [59, Theorem 1.6.6]. This factor 2 does not appear in Proposition 2.8.1 in this work, which is an analogue of [59, Remark 1.6.5]. In contrast, in Morel's case the extra factor 2 in L_M comes from [59, Remark 1.6.5].

After calculating $\mathrm{Tr}(v_h)$, the same arguments as on pp. 26-27 of [59] imply the modified version of [59, Theorem 1.7.1], where for $P = P_{n_1} \cap \cdots \cap P_{n_r}$ with P_{n_r} corresponding to the zero dimensional boundary data, we modify the definition of T_P ([59, p. 23]) as follows:

- Instead of letting $(\gamma_0, \gamma, \delta)$ run through the set $C'_{G_{n_r},j}$ defined in [59, Remark 1.6.5], we let it run through $C_{G_{n_r},j}$, but we add the compatibility condition between γ_0 and γ_L: We require that $\gamma_0 \in \mathrm{Cent}(\mathcal{X})$ if and only if $\gamma_L \in L(\mathbb{Q})^+$.

From this modified version of [59, Theorem 1.7.1], we easily deduce the special fiber version of (1.7.1) in this work. Namely we have proved the identity (1.7.1) for a sufficiently divisible, but with $\overline{\mathrm{Sh}_K}$ and Sh_K replaced by the mod p reductions of the integral models. Recall that these reductions are defined over $\mathbb{F}_{p^{a_0}}$ and not necessarily defined over $\mathbb{F}_p$, but the action of Frob_p^a makes sense as long as a is divisible by a_0. To prove the identity (1.7.1) as well as the asserted unramifiedness in the theorem, we apply [47, Theorem 4.18], which is valid because of the assumptions **(a)**, **(b)** above on the integral models. [7]

□

The following lemma constructs the integral models needed in the above proof of Theorem 1.7.2.

Lemma 2.7.3. *There exists a finite set of primes $\Sigma \supset \Sigma_0$ and a finite étale Galois ring extension R of $\mathbb{Z}[1/\Sigma]$ such that there exists a model $(\overline{\mathrm{Sh}_K})_R$ of $\overline{\mathrm{Sh}_K}$ over R, together with the models of the strata, satisfying the conditions modified from (1)-(7) on p. 8 of [59], and satisfying conditions **(a), (b)** in the above proof.*

Proof. We take an auxiliary Hodge type Shimura datum $\mathbf{D}$ for the abelian type Shimura datum $\mathbf{O}(V)$ as in [29], [31], and for simplicity of notation we assume the reflex field of $\mathbf{D}$ is $\mathbb{Q}$. For instance we may take $\mathbf{D}$ to be the Gspin Shimura datum (cf. [55, §3]). We use $\tilde{K}$ to denote a neat level for $\mathbf{D}$ and denote the associated Shimura variety (over $\mathbb{Q}$) by $\mathrm{Sh}_{\tilde{K}} := \mathrm{Sh}_{\tilde{K}}(\mathbf{D})$. In [54], integral models over $\mathbb{Z}_{(p)}$ of the toroidal compactifications of $\mathrm{Sh}_{\tilde{K}}$ are constructed via taking normalizations of the closures of $\mathrm{Sh}_{\tilde{K}}$ inside the integral models over $\mathbb{Z}_{(p)}$ of suitable toroidal compactifications of Siegel moduli spaces. Integral models over $\mathbb{Z}_{(p)}$ of the Baily-Borel compactification $\overline{\mathrm{Sh}_{\tilde{K}}}$, which we denote by $(\overline{\mathrm{Sh}_{\tilde{K}}})_p$, are constructed via the Proj construction applied to the *Hodge bundle* on the above-mentioned integral model of a toroidal compactification. For different p, the Hodge bundles are integral models of the same vector bundle on the generic fiber of the toroidal compactification. It follows that these integral models $(\overline{\mathrm{Sh}_{\tilde{K}}})_p$ glue together to a model $(\overline{\mathrm{Sh}_{\tilde{K}}})_{\mathbb{Z}[1/\Sigma_1]}$ over $\mathbb{Z}[1/\Sigma_1]$, for a large enough finite set of primes Σ_1. We may and will assume $\Sigma_1 \supset \Sigma_0$.

We may find a suitable $\tilde{K}$, enlarge Σ_1, and find a finite étale Galois ring extension

[7] In [47, §3] a special hypothesis is made on the relation between K and ℓ, but this can be easily removed by for instance considering the system of levels in [63, §4.9] instead of the system $\mathcal{H}(\ell^r), r > 0$ in the notation of [47, §3].

$\mathbb{Z}[1/\Sigma_1] \to R_1$, with $R_{1,\eta}$ a number field, such that over R_1 the analogue of statement (1) of [47, Proposition 2.4] is true. More precisely, all the connected components of $\mathrm{Sh}_K \otimes R_{1,\eta}$ and of $\mathrm{Sh}_{\tilde{K}} \otimes R_{1,\eta}$ are geometrically connected, and each connected component of $\mathrm{Sh}_K \otimes R_{1,\eta}$ is the quotient by the free action of a finite group of some connected component of $\mathrm{Sh}_{\tilde{K}} \otimes R_{1,\eta}$, such that all these finite group actions extend to free actions on the connected components of $(\overline{\mathrm{Sh}_{\tilde{K}}})_{R_1} := (\mathrm{Sh}_{\tilde{K}})_{\mathbb{Z}[1/\Sigma_1]} \otimes R_1$ that contain the connected components of $\mathrm{Sh}_{\tilde{K}} \otimes R_{1,\eta}$ in question. We then define $(\mathrm{Sh}_K)_{R_1}$ to be the union of the quotients of the aforementioned extended actions. It is a model of Sh_K over R_1.

Having constructed the pair $((\overline{\mathrm{Sh}_{\tilde{K}}})_{R_1}, (\mathrm{Sh}_K)_{R_1})$, we can now argue analogously as in the proof of [47, Proposition 2.4] to obtain a model $(\overline{\mathrm{Sh}_K})_{R_1}$ of $\overline{\mathrm{Sh}_K}$ over R_1, as well as models of its strata, on possibly enlarging Σ_1 and replacing R_1 by its corresponding localization. In fact, the crucial point used in that proof is the existence of a positive integer n such that the n-th power of the canonical line bundle on $\mathrm{Sh}_{\tilde{K}}$ extends to an ample line bundle on the canonical integral model of $\mathrm{Sh}_{\tilde{K}}$ over $\mathbb{Z}_{(p)}$, at a hyperspecial prime p. This integer n is given in the proof of [31, Proposition 4.6.28], and is clearly independent of p.

Since it is shown in [54] that for a hyperspecial prime p (for $\tilde{K}$) all the strata in $(\overline{\mathrm{Sh}_{\tilde{K}}})_p$ are finite quotients of hyperspecial canonical integral models, we know that every stratum in $(\overline{\mathrm{Sh}_K})_{R_1}$ when localized at any $R_{1,(p)}$ for $p \notin \Sigma_1$ gives the base change of the expected finite quotient of the hyperspecial canonical integral model.

With the above construction of $(\overline{\mathrm{Sh}_K})_{R_1}$, conditions **(a)** and **(b)** are satisfied (on possibly enlarging Σ_1 and localizing R_1). We then apply the argument in [59, Proposition 1.3.4] to further enlarge Σ_1 to get the set Σ, meanwhile replacing R_1 by its corresponding localization R, and obtain the desired model $(\overline{\mathrm{Sh}_K})_R$ over R. $\square$

In the following §2.8 and §2.9 we prove two results needed in the proof of Theorem 1.7.2. The notations and conventions in these two subsections only assume local validity and are different from the rest of this work.

2.8 An analogue of [59, Remark 1.6.5]

Let $M = G \times L$ be a product of two reductive groups over $\mathbb{Q}$. Assume G is a split torus and $\mathcal{X}$ is a Shimura datum for G (in the sense of Pink [62], so that $\mathcal{X}$ is a finite set) with reflex field $\mathbb{Q}$. Denote $\mathscr{L} := L(\mathbb{Q})/L(\mathbb{Q})^+$. Assume $|\mathcal{X}| = |\mathscr{L}| = 2$. Define an action of $L(\mathbb{Q})$ on $\mathcal{X}$ by letting $L(\mathbb{Q})^+$ act trivially and letting $\mathscr{L}$ act non-trivially. Let K_M be a neat compact open subgroup of $M(\mathbb{A}_f)$. Let

$$H := K_M \cap (G(\mathbb{A}_f)L(\mathbb{Q})) = K_M \cap (G(\mathbb{A}_f)M(\mathbb{Q})).$$

Assume that there exist $g_1 \in G(\mathbb{A}_f)$ and $l \in L(\mathbb{Q}) - L(\mathbb{Q})^+$ such that $g_1 l \in H$. Let

$$K := K_M \cap G(\mathbb{A}_f).$$

Let $\mathrm{Rep}_{G\times\mathscr{L}}$ be the category of algebraic representations of $G \times \mathscr{L}$ (necessarily all defined over $\mathbb{Q}$), where $\mathscr{L}$ is viewed as the constant group scheme $\mathbb{Z}/2\mathbb{Z}$.

Proposition 2.8.1. *Let ℓ be an arbitrary prime. Consider the Shimura variety $\mathrm{Sh}_K(G,\mathcal{X})$ and its finite quotient $\mathrm{Sh}_K(G,\mathcal{X})/H$. For almost all primes $p \neq \ell$, they have integral models over $\mathbb{Z}_{(p)}$ for which the following is true. The natural functor that goes from $\mathrm{Rep}_{G\times\mathscr{L}}$ to the category of $\mathbb{Q}_\ell$-sheaves on $\mathrm{Sh}_K(G,\mathcal{X})/H$ extends to a functor $\mathcal{F}_{\mathrm{int}}$ that goes from $\mathrm{Rep}_{G\times\mathscr{L}}$ to the category of lisse $\mathbb{Q}_\ell$-sheaves on the integral model of $\mathrm{Sh}_K(G,\mathcal{X})/H$. Keep the notations as in Theorem 1.6.1 of [59] with the following modifications:*

- *Consider $\mathscr{V} \in \mathrm{ObRep}_{\mathrm{G}\times\mathscr{L}}$ instead of $\mathscr{V} \in \mathrm{ObRep}_{\mathrm{G}}$.*
- *Replace $M^{K'}(G,\mathcal{X})$ by the integral model of $\mathrm{Sh}_K(G,\mathcal{X})/H$.*
- *Replace $\mathcal{F}^K$ by $\mathcal{F}_{\mathrm{int}}$.*

We have

$$T(j,g) = \sum_{(\gamma_0,\gamma,\delta)\in C_{G,j}} c(\gamma_0,\gamma,\delta)O_\gamma(f^p)TO_\delta(\phi_j^G)\widetilde{\mathrm{Tr}}(\gamma_0,\mathscr{V}). \tag{2.8.1}$$

Here the function f^p is defined using the compact open subgroup K_M/K_L of $G(\mathbb{A}_f)$, where $K_L := K_M \cap L(\mathbb{A}_f)$, and

$$\widetilde{\mathrm{Tr}}(\gamma_0,\mathscr{V}) := \begin{cases} \mathrm{Tr}(\gamma_0,\mathscr{V}), & \gamma_0 \in \mathrm{Cent}_{G(\mathbb{Q})}(\mathcal{X}) \\ \mathrm{Tr}(\gamma_0\times l_0,\mathscr{V}), & \gamma_0 \notin \mathrm{Cent}_{G(\mathbb{Q})}(\mathcal{X}), \end{cases}$$

where l_0 denotes the non-trivial element of $\mathscr{L}$.

Proof. The existence of $\mathcal{F}_{\mathrm{int}}$ for sufficiently large p follows from [82, Proposition 3.6] and the general construction of automorphic $\mathbb{Q}_\ell$-sheaves as recalled in [47, §3]. The rest of the proposition is an easy consequence of the description of the canonical integral models of zero dimensional Shimura varieties. cf. [63, §5.5]. □

2.9 An analogue of [59, Theorem 1.6.6]

In the following we state and prove an analogue of [59, Theorem 1.6.6], the latter being a special case of [18, 7.14 B].

Let L be a reductive group over $\mathbb{Q}$. Let $K, K' \subset L(\mathbb{A}_f)$ be neat open compact subgroups and fix $g \in L(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$. Let X_L be the homogeneous space for $L(\mathbb{R})$ as in [18, §7]. We call the double quotients

$$M^K_{\mathrm{shall}} := L(\mathbb{Q})^+\backslash X_L \times L(\mathbb{A}_f)/K$$

$$M^{K'}_{\mathrm{shall}} := L(\mathbb{Q})^+\backslash X_L \times L(\mathbb{A}_f)/K'$$

shallower locally symmetric spaces. Analogous to [59, p. 22], we have finite étale Hecke operators

$$T_1, T_g : M_{\text{shall}}^{K'} \to M_{\text{shall}}^{K}.$$

Let V be an algebraic representations of L over $\mathbb{C}$. Denote by $\mathcal{F}_{\text{shall}}^{K} V$ the associated sheaf on M_{shall}^{K}, namely the sheaf of local sections of

$$L(\mathbb{Q})^{+} \backslash (X_L \times L(\mathbb{A}_f) \times V/K) \to M_{\text{shall}}^{K}.$$

(cf. [59, p. 6].) The natural cohomological correspondence $T_g^{*} \mathcal{F}_s^{K} V \to T_1^{!} \mathcal{F}_s^{K} V$ gives rise to an endomorphism u_g on the compact support cohomology

$$R\Gamma_c^{\text{shall}}(K, V) := R\Gamma_c(M_{\text{shall}}^{K},\ \mathcal{F}_{\text{shall}}^{K} V)$$

analogously as in [59, Theorem 1.6.6]. Note that [59, Theorem 1.6.6] and [18] take different conventions on the definition of u_g. See [59, Remark 1.6.7] for more details on this. We follow the convention in [59] here.

We assume $|L(\mathbb{Q})/L(\mathbb{Q})^{+}| = 2$ and let l_0 denote the nontrivial element in $L(\mathbb{Q})/L(\mathbb{Q})^{+}$. We have a natural action of l_0 on M_{shall}^{K} by the diagonal action on $X_L \times L(\mathbb{A}_f)$. The sheaf $\mathcal{F}_{\text{shall}}^{K} V$ has a natural l_0-equivariant structure, and so l_0 induces an endomorphism of $R\Gamma_c^{\text{shall}}(K, V)$, which we still denote by l_0. It commutes with u_g.

Proposition 2.9.1. *We have*

$$\begin{aligned}\mathrm{Tr}(u_g, R\Gamma_c^{\mathrm{shall}}(K,V)) = 2\sum_{L'}(-1)^{\dim(A_{L'}/A_L)}(n_{L'}^L)^{-1} \\ \sum_\gamma \iota^{L'}(\gamma)^{-1}\chi(L'_\gamma)O_\gamma(f_{L'}^\infty)\left|D_{L'}^L(\gamma)\right|^{1/2}\mathrm{Tr}(\gamma,V).\end{aligned} \tag{2.9.1}$$

$$\begin{aligned}\mathrm{Tr}(u_g l_0, R\Gamma_c^{\mathrm{shall}}(K,V)) = 2\sum_{L'}(-1)^{\dim(A_{L'}/A_L)}(n_{L'}^L)^{-1} \\ \sum_\gamma \iota^{L'}(\gamma)^{-1}\chi(L'_\gamma)O_\gamma(f_{L'}^\infty)\left|D_{L'}^L(\gamma)\right|^{1/2}\mathrm{Tr}(\gamma,V).\end{aligned} \tag{2.9.2}$$

Here the notations are the same as in [59, Theorem 1.6.6], except for the following:

- *The symbols G and M in loc. cit. are replaced by L and L' here, respectively.*
- *In (2.9.1) (resp. (2.9.2)), the inner summation is over $L'(\mathbb{Q})$-conjugacy classes in $L'(\mathbb{Q}) \cap L(\mathbb{Q})^+$ (resp. $L'(\mathbb{Q}) - L(\mathbb{Q})^+$) that are $\mathbb{R}$-elliptic in $L'(\mathbb{R})$, instead of $L'(\mathbb{Q})$-conjugacy classes in $L'(\mathbb{Q})$ that are $\mathbb{R}$-elliptic.*

Proof. (2.9.1) follows from similar arguments as [18, §7]. In fact, the reductive Borel-Serre compactification and the weighted complexes on it are studied in [17] in the non-adelic setting, where one is allowed to pass to an arbitrary finite index subgroup of a chosen arithmetic subgroup. Hence by de-adelizing we can consider the reductive Borel-Serre compactification of M^K_{shall} and weighted complexes on it. Similarly to [18, §7.5], the reductive Borel-Serre compactification still has a stratification parametrized by the standard parabolic subgroups P, and the stratum parametrized by P is of the form

$$(L'(\mathbb{Q}) \cap L(\mathbb{Q})^+)\backslash[(N(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K) \times X_{L'}], \tag{2.9.3}$$

where L' is the standard Levi subgroup of P, N is the unipotent radical of P, and X_L is as in [18, §7.5]. The arguments in [18, §7] carry over, with the modified forms of the strata given by (2.9.3).

We explain some more details. We follow the notation of [18, §7], only replacing the symbols G and M with the symbols L and L'. In loc. cit., the fixed point counting first considers the spaces $\mathrm{Fix}(P, x_0, \gamma)$, where $P = L'N$ runs through the standard parabolic subgroups of L, x_0 runs through representatives of the double cosets $P(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K'$, and γ runs through conjugacy classes in $L'(\mathbb{Q})$. Each space $\mathrm{Fix}(P, x_0, \gamma)$ is given by

$$L'_\gamma(\mathbb{Q})\backslash(Y^\infty \times Y_\infty).$$

We refer the reader to [18, p. 523] for the definition of Y^∞ and Y_∞.

For us, recall that each stratum of the reductive Borel-Serre compactification is of the form (2.9.3), rather than $L'(\mathbb{Q})\backslash[(N(\mathbb{A}_f)\backslash L(\mathbb{A}_f)/K) \times X_M]$. Hence the same fixed point counting leads to consideration of spaces $\mathrm{Fix}'(P, x_0, \gamma)$, where P and x_0 run through the same sets as in the previous paragraph, but γ runs through conjugacy classes in $L'(\mathbb{Q}) \cap L(\mathbb{Q})^+$. Each space $\mathrm{Fix}'(P, x_0, \gamma)$ is of the form

$$\mathrm{Fix}'(P, x_0, \gamma) := (L'(\mathbb{Q}) \cap L^+(\mathbb{Q}))_\gamma\backslash(Y^\infty \times Y_\infty).$$

Let us call γ *of first kind* if $L'_\gamma(\mathbb{Q}) \subset L^+(\mathbb{Q})$. Otherwise we call γ *of second kind.* When γ is of first kind, there are precisely two $L'(\mathbb{Q})$-conjugacy classes in the $(L'(\mathbb{Q}) \cap L(\mathbb{Q})^+)$-conjugacy class of γ, and we have $\mathrm{Fix}'(P, x_0, \gamma) = \mathrm{Fix}(P, x_0, \gamma)$. When γ is of second kind, the $L'(\mathbb{Q})$-conjugacy class of γ is the same as the $(L'(\mathbb{Q}) \cap L(\mathbb{Q})^+)$-conjugacy class of γ, and $\mathrm{Fix}'(P, x_0, \gamma)$ is a two-fold covering of $\mathrm{Fix}(P, x_0, \gamma)$. In conclusion, the fixed point counting is equivalent to considering $\mathrm{Fix}''(P, x_0, \gamma)$, where P and x_0 run through the same sets as above, and γ runs through $L'(\mathbb{Q})$-conjugacy classes in $L'(\mathbb{Q}) \cap L(\mathbb{Q})^+$, with

$$\mathrm{Fix}''(P, x_0, \gamma) := \begin{cases} \text{disjoint union of two copies of } \mathrm{Fix}(P, x_0, \gamma), & \gamma \text{ of first kind} \\ \mathrm{Fix}'(P, x_0, \gamma), & \gamma \text{ of second kind.} \end{cases}$$

From the above definition, the space $\mathrm{Fix}''(P, x_0, \gamma)$ has its compact support Euler characteristic (cf. [18, §7.10, 7.11]) equal to twice of that of $\mathrm{Fix}(P, x_0, \gamma)$.

In the qualitative discussion in [18, §7.12], the contribution from $\mathrm{Fix}(P, x_0, \gamma)$ to the Lefschetz formula is a product of three factors (1) (2) (3), where factor (1) is the compact support Euler characteristic of $\mathrm{Fix}(P, x_0, \gamma)$. Analogously, for us the contribution from $\mathrm{Fix}''(P, x_0, \gamma)$ is also a product of the three analogous factors, where our factor (2) and (3) are identical to those in loc. cit., and our factor (1) is twice of the factor (1) in loc. cit., as we have already seen. Therefore, analogously to [18, (7.12.1)], we have the following expression for the Lefschetz formula

$$\sum_{P} \sum_{\gamma} 2(-1)^{\dim A_I / \dim A_{L'}} \left| L'_\gamma(\mathbb{Q}) / I(\mathbb{Q}) \right|^{-1} \chi(I) L_P(\gamma) O_\gamma(f_P) \tag{2.9.4}$$

where the notation is the same as loc. cit. except that M is replaced by L' and that the summation of γ is over $L'(\mathbb{Q})$-conjugacy classes in $L'(\mathbb{Q}) \cap L(\mathbb{Q})^+$ rather than conjugacy classes in $L'(\mathbb{Q})$.

Now the rest of the arguments in [18] deducing [18, Theorem 7.14 B] from [18, (7.12.1)] carry over to the analogue (2.9.4) of [18, (7.12.1)]. Also the elementary transformation from [18, Theorem 7.14 B] (applied to very positive weight profile, cf. [18, §7.17]) to the formula of [59, Theorem 1.6.6] carry over to imply (2.9.1).

We now prove (2.9.2). We claim that

$$\operatorname{Tr}(u_g, R\Gamma_c^{\mathrm{shall}}(K,V)) + \operatorname{Tr}(u_g l_0, R\Gamma_c^{\mathrm{shall}}(K,V)) = 2\operatorname{Tr}(u_g, R\Gamma_c(K,V)). \tag{2.9.5}$$

Here we have abusively used the notation u_g to also denote the endomorphism of $R\Gamma_c(K,V)$ induced by g. Once (2.9.5) is proved, the desired identity (2.9.2) follows from (2.9.1), (2.9.5), and the formula for $\operatorname{Tr}(u_g, R\Gamma_c(K,V))$ given by [59, Theorem 1.6.6].

We now prove (2.9.5). Denote

$$M^K := L(\mathbb{Q})\backslash X_L \times L(\mathbb{A}_f)/K.$$

Let π denote the double covering $\pi : M^K_{\mathrm{shall}} \to M^K$. Let $\mathcal{F}^K V$ be the sheaf of local sections of

$$L(\mathbb{Q})\backslash(X_L \times L(\mathbb{A}_f) \times V/K) \to M^K$$

(cf. [59, p. 6]). Because $\mathcal{F}^K_{\mathrm{shall}} V = \pi^* \mathcal{F}^K V$, we have

$$R\Gamma_c(M^K_{\mathrm{shall}}, \mathcal{F}^K_{\mathrm{shall}} V) = R\Gamma_c(M^K_{\mathrm{shall}}, \pi^* \mathcal{F}^K V). \tag{2.9.6}$$

Since π is a finite covering, we have

$$R\Gamma_c(M^K_{\mathrm{shall}}, \pi^* \mathcal{F}^K V) = R\Gamma_c(M^K, \pi_* \pi^* \mathcal{F}^K V). \tag{2.9.7}$$

For each character χ of the deck group $\Delta = \mathbb{Z}/2\mathbb{Z}$ of π, we let $\mathcal{F}_\chi$ be the local system given by the covering π and the character χ. Combining (2.9.6) (2.9.7) with the projection formula

$$\pi_* \pi^* \mathcal{F}^K V \cong \mathcal{F}^K V \otimes \bigoplus_{\chi : \Delta \to \mathbb{C}^\times} \mathcal{F}_\chi,$$

we obtain a decomposition

$$R\Gamma_c(M^K_{\text{shall}}, \mathcal{F}^K_{\text{shall}}V) \cong \bigoplus_{\chi} R\Gamma_c(M^K, \mathcal{F}^K V \otimes \mathcal{F}_\chi).$$

This decomposition is equivariant with respect to u_g, and the summand $R\Gamma_c(M^K, \mathcal{F}^K V \otimes \mathcal{F}_\chi)$ corresponds to the χ-eigenspace of Δ acting on $R\Gamma_c(M^K_{\text{shall}}, \mathcal{F}^K_{\text{shall}}V)$. Identity (2.9.5) follows.

□

3 Global and local quasi-split quadratic spaces, some generalities

In this section we review some generalities about quasi-split quadratic spaces over $\mathbb{Q}$ or $\mathbb{Q}_v$, where v is a place of $\mathbb{Q}$.

3.1 The odd case

Definition 3.1.1. Let $F = \mathbb{Q}$ or $\mathbb{Q}_v$, v a place of $\mathbb{Q}$. We say a dimension $d = 2m+1$ quadratic space (V_1, q_1) over F is *quasi-split* (or *split*, which is the same), if it is isomorphic to an orthogonal sum of m hyperbolic planes and a line.

In the local non-archimedean case, such spaces are characterized by the following relation between their Hasse invariant ϵ and discriminant δ:

$$\epsilon = (-1,-1)_v^{m(m-1)/2}(-1,\delta)_v^m = \begin{cases} (-1,\delta)_v^m, & v \neq 2 \\ (-1)^{m(m-1)/2}(-1,\delta)_v^m, & v = 2. \end{cases}$$

In the local archimedean case, such spaces are characterized by the condition that their signature is determined by the discriminant δ in the following way:

$$\begin{cases} \text{signature } &= (m+1, m), \qquad \text{if } \delta = (-1)^m \\ \text{signature } &= (m, m+1), \qquad \text{if } \delta = (-1)^{m+1}. \end{cases}$$

In the global case, such spaces are characterized by their localizations being quasi-split, at every place v. In both the local and the global cases, the isomorphism class of a quasi-split quadratic space of dimension d is determined by its discriminant, which can be an arbitrarily prescribed element of $F^\times/F^{\times,2}$.

Example 3.1.2. For a given dimension $d = 2m+1$ and discriminant $\delta \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$, up to isomorphism there exists at most one quadratic space over $\mathbb{Q}$ that is split over every non-archimedean place and has signature $(d-2, 2)$. When does such a space exist? By the product formula of the Hasse invariants we have $\epsilon_\infty = (-1, \delta)_\infty^m (-1)^{m(m-1)/2}$. On the other hand we have $\delta > 0$ and $\epsilon_\infty = (-1)^{2(2-1)/2} = -1$ by the signature condition. Therefore such a space exists if and only if $(-1)^{m(m-1)/2} = -1$, if and only if $m \equiv 2, 3 \mod 4$.

Remark 3.1.3. Let (V_1, q_1) be an arbitrary quadratic space over $\mathbb{Q}$ of dimension d. Then a sufficient condition for it to be quasi-split at a finite place $v \neq 2$ is that q_1 can be written in a diagonal form all of whose coefficients are v-units.

Proposition 3.1.4. *Let (V_1, q_1) be a quadratic space of dimension $d \geq 3$ odd over $F = \mathbb{Q}$ or $\mathbb{Q}_v$. Let $G_1 := \mathrm{SO}(V_1, q_1)$. Then (V_1, q_1) is split if and only if G_1 is split.*

Proof. This follows from [65, Proposition 2.14]. □

3.2 The even case

We remind the reader that we use the convention about the discriminant of an even dimensional quadratic space recalled in §0.7.

Definition 3.2.1. Let $F = \mathbb{Q}$ or $\mathbb{Q}_v$. Let $\delta \in F^\times/F^{\times,2}$ be arbitrary. We say a dimension $d = 2m$ quadratic space (V_1, q_1) over F is *quasi-split of discriminant* $\delta \in F^\times/F^{\times,2}$, if it is isomorphic to an orthogonal sum of $m-1$ hyperbolic planes and a two dimensional quadratic space of the form $(E = F(\sqrt{\delta}), \mathrm{N}_{E/F})$. (In case $\delta \in F^{\times,2}$, this last space is understood to be the vector space $F \oplus F\sqrt{\delta}$, with quadratic form $a \oplus b\sqrt{\delta} \mapsto a^2 - \delta b^2$.)

In the local non-archimedean case, such spaces are characterized by the following local invariants:

$$\text{discriminant} = \delta$$

$$\text{Hasse invariant} = (-1,-1)_v^{(m-1)(m-2)/2}(-1,-\delta)_v^{m-1} =$$

$$= \begin{cases} (-1,\delta)_v^{m-1}, & v \neq 2 \\ (-1)^{(m-1)(m-2)/2}(-1,-\delta)_v^{m-1}, & v = 2 \end{cases}$$

In the local archimedean case, such spaces are characterized by their signature being

$$\begin{cases} \text{signature } = (m+1, m-1), & \text{if } \delta < 0 \\ \text{signature } = (m, m), & \text{if } \delta > 0. \end{cases}$$

In the global case, such spaces are characterized by their localizations. In both the local and the global cases, the isomorphism class of a quasi-split quadratic space of dimension d is determined by its discriminant, which can be an arbitrarily prescribed element of $F^\times/F^{\times,2}$. Such a space is split, i.e. isomorphic to an orthogonal sum of m hyperbolic planes, if and only if $\delta \in F^{\times,2}$.

Remark 3.2.2. The orthogonal direct sum of $m-1$ hyperbolic planes and the plane

$$(F(\sqrt{\delta}), -\mathrm{N}_{F(\sqrt{\delta})/F})$$

also has discriminant δ, and its special orthogonal group is isomorphic to that of a quasi-split quadratic space of discriminant δ in the sense of Definition 3.2.1. Our preference towards one space over the other is nothing more than a convention. This convention is the same as Taïbi's convention in [73, §2.1]. Note that in the local non-archimedean case the Hasse invariant of the above space differs from the quasi-split one in the sense of Definition 3.2.1 by a factor of $(-1,\delta)_v$. Hence if δ is a v-unit the two quadratic spaces are isomorphic.

Definition 3.2.3. Let (V_1, q_1) be a quadratic space over $\mathbb{Q}_v$ of dimension d even, where v is an odd finite prime. Let $\delta \in \mathbb{Q}_v^\times/\mathbb{Q}_v^{\times,2}$. We say that (V_1, q_1) is *perfect quasi-split of discriminant* δ, if it is quasi-split of discriminant δ and δ has even valuation.

Remark 3.2.4. Let (V_1, q_1) be an arbitrary quadratic space over $\mathbb{Q}$ of dimension d even and discriminant δ. Then a sufficient condition for it to be perfect quasi-split at a finite place v is that q_1 can be written in a diagonal form all of whose coefficients are v-units.

Proposition 3.2.5. *Let (V_1, q_1) be a quadratic space of even dimension $d \geq 4$ over $\mathbb{Q}_v$, where v is an odd prime. Let $G_1 = \mathrm{SO}(V_1, q_1)$. Then (V_1, q_1) is perfect quasi-split if and only if G_1 is unramified.*

Proof. Suppose (V_1, q_1) is perfect quasi-split. If (V_1, q_1) is split, i.e. an orthogonal direct sum of hyperbolic planes, then a hyperbolic basis spans a unimodular lattice. Assume (V_1, q_1) is quasi-split and non-split. Let $\delta \in \mathbb{Z}_v^\times$ be a representative of the discriminant. By definition, we can find a $\mathbb{Q}_v$-basis

$$e_1, \cdots, e_{m-1}, e_1', \cdots, e_{m-1}', f_1, f_2$$

of V_1, such that

$$[e_i, e_j] = [e'_i, e'_j] = [e_i, f_j] = [e'_i, f_j] = 0$$
$$[e_i, e'_j] = \delta_{ij} \tag{3.2.1}$$
$$[f_1, f_1] = 1, [f_2, f_2] = -\delta, [f_1, f_2] = 0.$$

They span a unimodular $\mathbb{Z}_v$-lattice, because $\delta \in \mathbb{Z}_v^\times$. We see that in all cases V_1 contains a unimodular $\mathbb{Z}_v$-lattice. The special orthogonal group of that lattice is a reductive model over $\mathbb{Z}_v$ for G_1. Hence G_1 is unramified.

Conversely, assume G_1 is unramified. Then G_1 is quasi-split over $\mathbb{Q}_v$ and split over an unramified extension of $\mathbb{Q}_v$. If G_1 is split, then [65, Proposition 2.14] implies that (V_1, q_1) is split, and a fortiori perfect quasi-split.

Assume G_1 is non-split. Then by the abstract classification of quasi-split reductive groups we know that G_1 is split over a unique (up to $\mathbb{Q}_v$-algebra isomorphisms) quadratic extension E of $\mathbb{Q}_v$. Since G_1 is unramified, E is unramified over $\mathbb{Q}_v$. By loc. cit., $V_1 \otimes E$ as a quadratic space over E is split. In particular, $\delta \in E^{\times,2}$. We conclude that δ has even v-adic valuation.

By the abstract classification we know that the $\mathbb{Q}_v$-split rank of G_1 is $d/2 - 1$. Hence by loc. cit. we know that V_1 is an orthogonal direct sum of $d/2 - 1$ hyperbolic planes and a two dimensional quadratic space P. The discriminant of P is δ and P is not split. It follows that P has to be isomorphic to $(\mathbb{Q}_v(\sqrt{\delta}), \mathrm{N}_{\mathbb{Q}_v(\sqrt{\delta})/\mathbb{Q}_v})$. □

4 Calculation at infinity

4.1 Levi subgroups and elliptic tori

Definition 4.1.1. We say a reductive group over $\mathbb{R}$ is *cuspidal* if it contains an elliptic maximal torus. We say a reductive group over $\mathbb{Q}$ is *cuspidal* if its base change to $\mathbb{R}$ is cuspidal.

Remark 4.1.2. Our definition of cuspidality for a reductive group over $\mathbb{Q}$ appears to be non-standard. The standard definition for a reductive group G_1 over $\mathbb{Q}$ to be cuspidal seems to be that $(G_1/A_{G_1})_{\mathbb{R}}$ has an anisotropic maximal torus over $\mathbb{R}$, where A_{G_1} is the maximal $\mathbb{Q}$-split subtorus of Z_{G_1}. Since $(A_{G_1})_{\mathbb{R}}$ could be smaller than $A_{G_{1,\mathbb{R}}}$, G_1 might not be cuspidal in this sense even if $G_{1,\mathbb{R}}$ contains a maximal elliptic torus. Such phenomena do not happen to groups considered in this work, since they will all be a product of a semi-simple group and a GL_j.

We pass to a local setting over $\mathbb{R}$. The symbols V, V_i, G, P_S, M_S will now denote the base change over $\mathbb{R}$ of the corresponding objects considered in 1.1. We keep the choice of the flag (1.1.1) (now considered as over $\mathbb{R}$) and a splitting of it as in 1.1. Moreover we choose $e_i, e_i', i = 1, 2$, as in 1.1. With these choices, we identify

(4.1.1)

$$M_1 \cong \mathrm{GL}_2 \times \mathrm{SO}(V_2^{\perp}/V_2), \quad M_2 \cong \mathbb{G}_m \times \mathrm{SO}(V_1^{\perp}/V_1), \quad M_{12} \cong \mathbb{G}_m^2 \times \mathrm{SO}(V_2^{\perp}/V_2),$$

as algebraic groups over $\mathbb{R}$. We will write points of M_S as (g, h) or (g_1, g_2, h) according to the above decomposition.

Note that $V_2^{\perp}/V_2$ and $V_1^{\perp}/V_1$ are quadratic spaces over $\mathbb{R}$ of signatures $(n-2, 0)$ and $(n-1, 1)$, respectively. Among the three groups (4.1.1), M_1 and M_{12} are always cuspidal, whereas M_2 is cuspidal if and only if d is odd.

Recall from Definition 1.1.1 that we denote $W_2 := V_2^{\perp}/V_2$. We fix an ($\mathbb{R}$-)elliptic maximal torus T_{W_2} of the anisotropic group $\mathrm{SO}(W_2) \cong \mathrm{SO}(n-2)$. Explicitly, we may choose a decomposition of W_2, which is of signature $(n-2, 0)$, into an orthogonal direct sum of positive definite planes (plus a positive definite line in the odd case), and take T_{W_2} to be the embedded product of copies of $\mathrm{U}(1) \cong \mathrm{SO}(2, 0) \cong \mathrm{SO}(0, 2)$ according to this decomposition.

We then obtain an elliptic maximal torus T_S of M_S for $S = \{1\}$ or $\{1, 2\}$ as follows:

$$T_1 := T^{\mathrm{std}}_{\mathrm{GL}_2} \times T_{W_2}, \quad T_{12} := \mathbb{G}_m^2 \times T_{W_2},$$

where $T^{\mathrm{std}}_{\mathrm{GL}_2} \subset \mathrm{GL}_2$ is defined by

$$T^{\mathrm{std}}_{\mathrm{GL}_2}(R) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} | a, b \in R, a^2 + b^2 \neq 0 \right\},$$

for any $\mathbb{R}$-algebra R.

In the odd case we also choose an elliptic maximal torus T_2 of M_2 of the form $T_2 = \mathbb{G}_m \times T_{W_1}$, where T_{W_1} is an elliptic maximal torus of $\mathrm{SO}(W_1) = \mathrm{SO}(V_1^{\perp}/V_1)$.

4.2 Root data

We define a split maximal torus T' of $G_{\mathbb{C}}$ as follows. We choose a maximal isotropic subspace $V_{\max}$ of $V_{\mathbb{C}}$ *containing* $V_{2,\mathbb{C}}$. Let

$$m = \dim V_{\max} = \lfloor \frac{d}{2} \rfloor.$$

Fix a splitting of the flag

$$0 \subset V_{1,\mathbb{C}} \subset V_{2,\mathbb{C}} \subset V_{\max} \subset V_{\max}^{\perp} \subset V_{2,\mathbb{C}}^{\perp} \subset V_{1,\mathbb{C}}^{\perp} \subset V_{\mathbb{C}} \tag{4.2.1}$$

that is *compatible with the chosen splitting* of (1.1.1). Fix a $\mathbb{C}$-basis

$$e_3, e_4, \cdots, e_m$$

of $V_{\max}/V_{2,\mathbb{C}}$. Using the splitting of 4.2.1, we think of them as vectors of $V_{\max}$. Then $e_1, e_2, \cdots, e_m$ form a basis of $V_{\max}$. The bilinear pairing on $V_{\mathbb{C}}$ identifies $V_{\mathbb{C}}/V_{\max}^{\perp}$ with the $\mathbb{C}$-linear dual of $V_{\max}$, and we let $e'_1, \cdots, e'_m$ be the dual basis of $V_{\mathbb{C}}/V_{\max}^{\perp}$, which would start with e'_1 and e'_2 considered in §1.1. Via the chosen splitting of (4.2.1) we regard e'_i as vectors of $V_{\mathbb{C}}$. If $V_{\max}^{\perp}/V_{\max} \neq 0$ (that is, in the odd case), it is one dimensional and we choose a basis e_0 for it such that $q(e_0) = 1$, again regarded as a vector in V. Define T' to be the image of the following embedding:

$$\mathbb{G}_{m,\mathbb{C}}^m \to G_{\mathbb{C}},$$

$$(c_1, \cdots, c_m) \mapsto \begin{pmatrix} e_i \mapsto c_i e_i \\ e'_i \mapsto c_i^{-1} e'_i \\ (e_0 \mapsto e_0) \end{pmatrix}$$

We identify T' with $\mathbb{G}_{m,\mathbb{C}}^m$ using the above map and write points of T' as $(c_1, \cdots, c_m)$.

By construction, $T' \subset M_{\mathbb{C}}$ for any standard $M \in \{M_1, M_2, M_{12}\}$. Moreover, the factor $\mathbb{G}_{m,\mathbb{C}}^2$ of $T_{12,\mathbb{C}}$ is contained inside T' as the first two factors of $T' \cong \mathbb{G}_{m,\mathbb{C}}^m$. We fix an element $g_{W_2} \in \mathrm{SO}(W_2)$ such that $\mathrm{Int}\, g_{W_2}$ maps the torus $T_{W_2,\mathbb{C}} \subset \mathrm{SO}(W_2)_{\mathbb{C}} \subset G_{\mathbb{C}}$ chosen before into T'. Then we have isomorphisms:

(4.2.2)
$$\iota_{12} := \mathrm{Int}(1, g_{W_2}) : T_{12,\mathbb{C}} \xrightarrow{\sim} T'$$
$$\iota_1 := \mathrm{Int}(\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, g_{W_2}) : T_{1,\mathbb{C}} \xrightarrow{\sim} T'.$$

In particular the second map sends $(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, 1)$ to $(a + bi, a - bi, 1, \cdots, 1)$.

Denote the standard characters of $\mathbb{G}_{m,\mathbb{C}}^m \cong T'$ by $\epsilon_1, \cdots, \epsilon_m$, and the standard cocharacters by $\epsilon_1^{\vee}, \cdots, \epsilon_m^{\vee}$. We transport them to $T_{S,\mathbb{C}}$ using ι_S, for $S = \{1, 2\}$ or $\{1\}$, and retain the same notation.

The odd case.

The roots of $(G_{\mathbb{C}}, T')$ are given by

$$\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, i \neq j,$$

and the coroots are given by

$$\pm\epsilon_i^\vee \pm \epsilon_j^\vee, \pm 2\epsilon_i^\vee, i \neq j.$$

Among the roots, resp. coroots, the ones that are real with respect to T_1 are $\pm(\epsilon_1 + \epsilon_2)$, resp. $\pm(\epsilon_1^\vee + \epsilon_2^\vee)$. We will denote the resulting root datum by

$$\Phi_1 = (\mathbb{Z}(\epsilon_1 + \epsilon_2), \mathbb{Z}(\epsilon_1^\vee + \epsilon_2^\vee), R_1, R_1^\vee) \subset \Phi(G_{\mathbb{C}}, T').$$

Among the roots, resp. coroots, the ones that are real with respect to T_{12} are

$$\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_1 \pm \epsilon_2,$$

resp.

$$\pm 2\epsilon_1^\vee, \pm 2\epsilon_2^\vee, \pm\epsilon_1^\vee \pm \epsilon_2^\vee.$$

We will denote the resulting root datum by

$$\Phi_{12} = (\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2, \mathbb{Z}\epsilon_1^\vee \oplus \mathbb{Z}\epsilon_2^\vee, R_{12}, R_{12}^\vee) \subset \Phi(G_{\mathbb{C}}, T').$$

The even case.

The roots of $(G_{\mathbb{C}}, T')$ are given by

$$\pm\epsilon_i \pm \epsilon_j, i \neq j,$$

and the coroots are given by

$$\pm\epsilon_i^\vee \pm \epsilon_j^\vee, i \neq j.$$

Among the roots, resp. coroots, the ones that are real with respect to T_1 are $\pm(\epsilon_1 + \epsilon_2)$, resp. $\pm(\epsilon_1^\vee + \epsilon_2^\vee)$. We will denote the resulting root datum by

$$\Phi_1 = (\mathbb{Z}(\epsilon_1 + \epsilon_2), \mathbb{Z}(\epsilon_1^\vee + \epsilon_2^\vee), R_1, R_1^\vee) \subset \Phi(G_{\mathbb{C}}, T').$$

Among the roots, resp. coroots, the ones that are real with respect to T_{12} are

$$\pm\epsilon_1 \pm \epsilon_2,$$

resp.

$$\pm\epsilon_1^\vee \pm \epsilon_2^\vee.$$

We will denote the resulting root datum by

$$\Phi_{12} = (\mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2, \mathbb{Z}\epsilon_1^\vee \oplus \mathbb{Z}\epsilon_2^\vee, R_{12}, R_{12}^\vee) \subset \Phi(G_{\mathbb{C}}, T').$$

In the odd case, the root systems in $\Phi(G_{\mathbb{C}}, T_{\mathbb{C}})$, Φ_1, Φ_{12} are of type B_m, A_1, B_2 respectively. *In the even case*, the root systems in $\Phi(G_{\mathbb{C}}, T_{\mathbb{C}})$, Φ_1, Φ_{12} are of type $D_m, A_1, A_1 \times A_1$ respectively.

4.3 Elliptic endoscopic data for G

The discussion in this subsection is valid for a general quadratic space (V, q) over $\mathbb{R}$, with no assumption on the signature. In fact we will need such generality too. We thus temporarily drop the assumption that the signature of (V, q) is $(d-2, 2)$ and consider a general quadratic space (V, q) over $\mathbb{R}$, with discriminant $\delta \in \mathbb{R}^\times/\mathbb{R}^{\times,2} \cong \{\pm 1\}$ and of dimension d. Let

$G = \mathrm{SO}(V, q)$ as an algebraic group over $\mathbb{R}$. Let $m = \lfloor d/2 \rfloor$ be the absolute rank of G.

The goal of this subsection is to recall the classification of elliptic endoscopic data of G in terms of smaller special orthogonal groups. For each elliptic endoscopic datum $(H, \mathcal{H}, s, \eta)$, we also fix an explicit isomorphism $\mathcal{H} \cong {}^L H$. In the following we shall exhibit the well known way of doing this explicitly, for G a special orthogonal group. Thus we do not have to consider z-extensions for the theory of endoscopy (which is necessary in general) even though G^{der} is not simply connected in our case.

Remark 4.3.1. In this work, we will always consider endoscopic data in terms of explicitly chosen representatives in the isomorphism classes. For each such representative $(H, \mathcal{H}, s, \eta)$, we always explicitly choose an isomorphism $\mathcal{H} \cong {}^L H$. Thus in the terminology of [26], we upgrade each endoscopic datum to an *extended endoscopic datum*.

In the following we follow the notation and conventions of [81].

4.3.2 Fixing the quasi-split inner form

Fix a quadratic space $(\underline{V}, \underline{q})$ over $\mathbb{R}$ that is quasi-split and has the same dimension and discriminant as (V, q) (cf. Definition 3.1.1 and Definition 3.2.1). Recall from §3 that the isomorphism class of $(\underline{V}, \underline{q})$ is uniquely determined. Also recall from Remark 3.2.2 that in the even case we do not consider the quadratic space of signature $(d/2 - 1, d/2 + 1)$ as quasi-split.

Definition 4.3.3. In the even case, we fix an orientation $o_{\underline{V}}$ on $\underline{V}$ once and for all.

Definition 4.3.4. We fix an isomorphism of quadratic spaces over $\mathbb{C}$

$$\phi : (V, q) \otimes \mathbb{C} \xrightarrow{\sim} (\underline{V}, \underline{q}) \otimes \mathbb{C},$$

such that the following is true: There exists an orthogonal basis $v_1, \cdots, v_d$ of V, and an orthogonal basis $\underline{v}_1, \cdots, \underline{v}_d$ for $\underline{V}$, such that for each $1 \leq i \leq d$, we have $q(v_i), \underline{q}(\underline{v}_i) \in \{\pm 1\}$, and $\phi(v_i) = \epsilon_i \underline{v}_i$ for some $\epsilon_i \in \left\{1, \sqrt{-1}\right\}$.

Define $G^* := \mathrm{SO}(\underline{V}, \underline{q})$. Using ϕ we get an isomorphism

$$\psi = \psi_\phi : G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}, \quad g \mapsto \phi g \phi^{-1}.$$

Define a function

$$u : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{GL}(\underline{V} \otimes \mathbb{C}) \quad \rho \mapsto {}^\rho\phi\phi^{-1}.$$

Then u lands in $\mathrm{O}(\underline{V})(\mathbb{C})$. Moreover, since V and $\underline{V}$ have the same discriminant, in fact u lands in $G^*(\mathbb{C})$.

Note that we have

$${}^\rho\psi\psi^{-1} = \mathrm{Int}(u(\rho)) : G^*_{\mathbb{C}} \to G^*_{\mathbb{C}}.$$

It follows that ψ is an inner twisting.

Remark 4.3.5. Abstractly speaking, fixing ϕ and getting the pair (ψ, u) as above means that we have realized G as a *pure inner form* of G^* in the sense of Vogan [78]. This is essential in the discussion of transfer factors in §4.5, §4.6, §4.7 below. See the introduction of [27] for more historical remarks on this notion and other related notions, which have all been generalized to the notion of rigid inner forms by Kaletha loc. cit..

Definition 4.3.6. In the even case, define the orientation o_V on V by the orientation $o_{\underline{V}}$ on $\underline{V}$ and the $\mathbb{R}$-linear isomorphism[8]

$$\wedge^d\phi : \wedge^d V \xrightarrow{\sim} \wedge^d \underline{V}.$$

[8]Here $\wedge^d\phi$ is defined over $\mathbb{R}$ because V and $\underline{V}$ have the same discriminant.

4.3.7 Fixing the L-group

First we define some algebraic groups over $\mathbb{C}$. These groups shall be identified with their $\mathbb{C}$-points. Let $N \in \mathbb{Z}_{\geq 1}$. Let $(\hat{e}_k)_{1\leq k\leq N}$ be the standard basis of $\mathbb{C}^N$. Define I_N^- to be the $N\times N$ matrix whose (i,j)-th entry is

$$(I_N^-)_{ij} = \begin{cases} (-1)^i, & i+j=N+1 \\ 0, & \text{otherwise.} \end{cases}$$

When N is odd, define $I_N^+ := I_N^-$. When N is even, define I_N^+ to be the $N\times N$ matrix whose (i,j)-th entry is

$$(I_N^+)_{ij} = \begin{cases} (-1)^i, & i+j=N+1, i\leq N/2 \\ (-1)^{i+1}, & i+j=N+1, i> N/2 \\ 0, & \text{otherwise.} \end{cases}$$

Thus I_N^+ defines a symmetric bilinear form on $\mathbb{C}^N$, and when N is even I_N^- defines a symplectic from on $\mathbb{C}^N$. We use them to form the groups $\mathrm{SO}_N(\mathbb{C})$, and, when N even, $\mathrm{Sp}_N(\mathbb{C})$. By convention, $\mathrm{SO}_0(\mathbb{C}) = \mathrm{Sp}_0(\mathbb{C}) = \mathrm{GL}_0(\mathbb{C}) = \{1\}$.

Definition 4.3.8. In the matrix group $\mathrm{Sp}_{d-1}(\mathbb{C})$ (resp. $\mathrm{SO}_d(\mathbb{C})$), we fix once and for all a Borel pair $(\mathcal{T},\mathcal{B})$, together with a trivialization $\mathcal{T} \xrightarrow{\sim} \mathbb{G}_m^m$, as follows. Let $N=d-1$ (resp. $N=d$) and let $m=\lfloor d/2\rfloor$. Consider the embedding $\mathbb{G}_m^m \hookrightarrow \mathrm{Sp}_N(\mathbb{C})$ (resp. $\mathbb{G}_m^m \hookrightarrow \mathrm{SO}_N(\mathbb{C})$) such that the i-th $\mathbb{G}_m$ acts by weight 1 on $\hat{e}_i$ and weight -1 on $\hat{e}_{N+1-i}$ and weight zero on $\hat{e}_j$ for $j\in\{1,2,\cdots,N\}-\{i,N+1-i\}$. Define $\mathcal{T}$ to be the image of this embedding. In other words, $\mathcal{T}$ is the diagonal torus. The resulting root datum on $(X^*(\mathbb{G}_m^m), X_*(\mathbb{G}_m^m)) = (\mathbb{Z}^m,\mathbb{Z}^m)$ is dual to the standard type B (resp. D) rood datum. Define $\mathcal{B}$ corresponding to the natural order. In the following we also call $(\mathcal{T},\mathcal{B})$ the *natural* Borel pair.

We now define the L-group of G. In the odd case we take ${}^LG = \mathrm{Sp}_{d-1}(\mathbb{C})\times W_{\mathbb{R}}$. If d is even and $\delta > 0$, we take ${}^L G = \mathrm{SO}_d(\mathbb{C}) \times W_{\mathbb{R}}$.

In the remaining case, namely when d is even and $\delta < 0$, we take ${}^L G = \mathrm{SO}_d(\mathbb{C}) \rtimes W_{\mathbb{R}}$ with a non-trivial action as follows. The subgroup $W_{\mathbb{C}} = \mathbb{C}^\times$ of $W_{\mathbb{R}}$ acts trivially. For $\rho \in W_{\mathbb{R}} - W_{\mathbb{C}}$, it acts on $\mathrm{SO}_d(\mathbb{C})$ by conjugation by the permutation matrix on $\mathbb{C}^d$ that switches $\hat{e}_{d/2}$ and $\hat{e}_{d/2+1}$ and fixes all the other $\hat{e}_i$'s. We remark that this automorphism of $\mathrm{SO}_d(\mathbb{C})$ is non-inner, and also in this case the map $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Out}(G_{\mathbb{C}}) \cong \mathbb{Z}/2\mathbb{Z}$ that determines G as an outer form of the split special orthogonal group is non-trivial. This case is the only case where the quasi-split inner form G^* is not split.

As usual, we set $\hat{G}$ to be the identity component of ${}^L G$. Thus $\hat{G} = \mathrm{Sp}_{d-1}(\mathbb{C})$ or $\mathrm{SO}_d(\mathbb{C})$ according to whether d is odd or even. We think of $\hat{G}$ as being equipped with the L-action mentioned above that was used to form ${}^L G$.

Strictly speaking, to regard $\hat{G}$ as the dual group to G^*[9], we need to specify a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant isomorphism between the canonical based root datum of $\hat{G}$ with the dual of the canonical based root datum of G^*. What amounts to the same is that, for one Borel pair $(\mathcal{T}, \mathcal{B})$ in $\hat{G}$ and one Borel pair (T, B) in G^*, we need to specify a suitable isomorphism $\hat{T} \xrightarrow{\sim} \mathcal{T}$. Recall from Definition 4.3.8 that we have already fixed a Borel pair $(\mathcal{T}, \mathcal{B})$ in $\hat{G}$, together with a trivialization $\mathcal{T} \xrightarrow{\sim} \mathbb{G}_m^m$.

Definition 4.3.9. A *hyperbolic decomposition (h.d.)* of $\underline{V}$ is an (ordered) m-tuple of oriented hyperbolic planes $V_1, \cdots, V_m$ in V, plus a line in V in the odd case, such that V is the orthogonal direct sum of them. A *near-hyperbolic decomposition (n.h.d.)* of $\underline{V}$ is an (ordered) $(m-1)$-tuple of oriented hyperbolic planes $V_1, \cdots, V_{m-1}$ in V, together with an oriented definite plane V_m in V, such that V is the orthogonal direct sum of them.

Definition 4.3.10. A *parametrized split maximal torus* of G^* is a split maximal torus T of G^* together with an isomorphism $\mathbb{G}_m^m \xrightarrow{\sim} T$. A *parametrized near-split maximal torus* of G^* is a maximal torus T of G^*, together with an isomorphism $\mathbb{G}_m^{m-1} \times \mathrm{U}(1) \xrightarrow{\sim} T$.

[9] namely to upgrade $\hat{G}$ to a "dual group datum".

Recall that we need to specify an identification $\hat{T} \xrightarrow{\sim} \mathcal{T}$ for a given Borel pair (T, B) in G^*.

We first treat the odd case. In this case $\underline{V}$ is split, and we can choose an h.d. of $\underline{V}$. This gives rise to an embedding $\mathbb{G}_m^m \hookrightarrow G^*$, whose $G^*(\mathbb{R})$-conjugacy class is independent of the choice of the h.d.. The embedding is given as follows. The i-th copy of $\mathbb{G}_m$ acts trivially on all the hyperbolic planes and the line except the i-th hyperbolic plane, where it acts with weight 1 on v and weight -1 on w, where (v, w) is an oriented basis of the i-th plane such that $\underline{q}(v) = \underline{q}(w) = 0, [v, w]_{\underline{q}} = 1$. We thus get a parametrized split maximal torus T of G^*. The resulting root datum on $(X^*(\mathbb{G}_m^m), X_*(\mathbb{G}_m^m)) = (\mathbb{Z}^m, \mathbb{Z}^m)$ is the standard type B root datum. We choose B corresponding to the natural order. We have now specified (B, T) and $(\mathcal{B}, \mathcal{T})$, and we ought to specify an isomorphism $\hat{T} \xrightarrow{\sim} \mathcal{T}$. We have already identified both T and $\mathcal{T}$ with $\mathbb{G}_m^m$, and we now choose the isomorphism $\hat{T} \xrightarrow{\sim} \mathcal{T}$ to be the resulting one.

Now we treat the even case with $\underline{V}$ split. Again we choose an h.d. of $\underline{V}$ and from that get a parametrized split maximal torus of G^*. The difference with the odd case is that, such constructions give rise to two $G^*(\mathbb{R})$-conjugacy classes of embeddings $\mathbb{G}_m^{d/2} \hookrightarrow G^*$, corresponding to the two resulting orientations on $\underline{V}$. The orientation $o_{\underline{V}}$ fixed in Definition 4.3.3 allows us single out one from the two $G^*(\mathbb{R})$-conjugacy classes that is compatible with $o_{\underline{V}}$. Now we proceed similarly as the odd case. We get a parametrized split maximal torus T, and the resulting root datum on $(\mathbb{Z}^m, \mathbb{Z}^m)$ is the standard type D root datum. We choose B corresponding to the natural order. We now choose the isomorphism $\hat{T} \xrightarrow{\sim} \mathcal{T}$ to be the one that results from having identified both T and $\mathcal{T}$ with $\mathbb{G}_m^m$.

Finally we treat the even case with $\underline{V}$ non-split. In this case we choose an n.h.d. of $\underline{V}$ and get a parametrized near-split maximal torus $\mathbb{G}_m^{m-1} \times \mathrm{U}(1) \xrightarrow{\sim} T \subset G^*$, such that $\mathbb{G}_m^{m-1}$ acts on the oriented hyperbolic planes as before, and $\mathrm{U}(1)$ acts on the oriented definite plane by rotation. Again, the orientation $o_{\underline{V}}$ on $\underline{V}$ fixed in Definition 4.3.3 singles out one $G^*(\mathbb{R})$-conjugacy class of such embeddings. We identify $X^*(\mathrm{U}(1))$ with $\mathbb{Z}$, on which complex conjugation acts as multiplication by -1. We can now proceed in the same way as the even split case. We note that according to the same construction as in the even split case, the factor $\mathrm{U}(1)$ of T is paired with the last $\mathbb{G}_m$-factor of $\mathcal{T} = \mathbb{G}_m^{d/2}$, namely the factor that acts with weight 1 on $\hat{e}_{d/2}$.

4.3.11 The elliptic endoscopic data

We choose representatives of isomorphism classes of elliptic endoscopic data for G as follows. Such a datum will be of the form

$$(H, s, \eta) = (H, \mathcal{H}, s, \eta),$$

where

- H is a quasi-split reductive group over $\mathbb{R}$,
- $\mathcal{H} = {}^L H$ is the L-group of H (cf. Remark 4.3.1),
- s is a semi-simple element of $\hat{G}$,
- η is an L-embedding $\eta : {}^L H \to {}^L G$.

In fact, H will be a product $H = H^+ \times H^-$ of groups $H^\pm$ of the form $\mathrm{SO}(V^\pm, q^\pm)$ for certain quasi-split quadratic spaces $(V^\pm, q^\pm)$ over $\mathbb{R}$. We represent the L-groups ${}^L H^\pm$ of $H^\pm$ in the same way as the L-group of G described above (where we choose the identity $H^\pm \to H^\pm$ as the inner twisting to the quasi-split inner form, and in case $V^\pm$ is even dimensional we fix an orientation on $V^\pm$ and use that to identify the explicitly defined group ${}^L H^\pm$ as the L-group of $H^\pm$), and we take the L-group ${}^L H$ of H to be the fiber product of ${}^L H^+$ and ${}^L H^-$ over $W_\mathbb{R}$. In particular ${}^L H$ is a certain semi-product

$${}^L H = (\hat{H}^+ \times \hat{H}^-) \rtimes W_\mathbb{R}.$$

For $w \in W_{\mathbb{R}}$, we write

$$\eta(w) = (\rho(w), w) \in {}^L G = \hat{G} \rtimes W_{\mathbb{R}},$$

where ρ is a certain map $W_{\mathbb{R}} \to \hat{G}$. To specify the map η it suffices to specify $\eta|_{\hat{H}}$ and ρ. The element $s \in \hat{G}$ will always be a diagonal matrix, with only ± 1's on the diagonal. We write $s = \mathrm{diag}(s_1, \cdots, s_{d-1})$ or $\mathrm{diag}(s_1, \cdots, s_d)$, when d is odd or even respectively.

The odd case.

Consider $d^+, d^- \in \mathbb{Z}_{\geq 0}$ both odd and such that $d^+ + d^- = d + 1$. Denote $m^\pm := \lfloor d^\pm/2 \rfloor$. Take $(V^\pm, q^\pm)$ to be quasi-split (i.e. split) quadratic spaces of dimension $d^\pm$ and trivial discriminant (cf. Definition 3.1.1). Define s by

$$s_k := \begin{cases} 1, & m^- + 1 \leq k \leq d - m^- - 1 \\ -1, & \text{otherwise.} \end{cases}$$

We have ${}^L H = (\hat{H}^+ \times \hat{H}^-) \times W_{\mathbb{R}}$. Define $\eta : {}^L H \to {}^L G$ to be the map glued from the identity on $W_{\mathbb{R}}$ and the following map on $\hat{H} = \hat{H}^+ \times \hat{H}^-$:

$$\eta|_{\hat{H}^+ \times \hat{H}^-} : \hat{H}^+ \times \hat{H}^- = \mathrm{Sp}_{d^+-1}(\mathbb{C}) \times \mathrm{Sp}_{d^--1}(\mathbb{C}) \to \hat{G} = \mathrm{Sp}_{d-1}(\mathbb{C})$$

given by the identification

$$\mathbb{C}^{d^+-1} \times \mathbb{C}^{d^--1} \xrightarrow{\sim} \mathbb{C}^{d-1}$$

$$(\hat{e}_k, 0) \mapsto \hat{e}_{k+m^-}$$

$$(0, \hat{e}_k) \mapsto \begin{cases} \hat{e}_k, & k \leq m^- \\ \hat{e}_{k+d^+-1}, & m^- + 1 \leq k \leq d^- - 1. \end{cases}$$

The above construction gives rise to an (extended) elliptic endoscopic datum

$$\mathfrak{e}_{d^+,d^-} = (H^+ \times H^-, s, \eta)$$

for G, which is well defined up to isomorphism. Every elliptic endoscopic datum for G, up to isomorphism, arises in this way, and two pairs (d^+, d^-) and (d_1^+, d_1^-) give rise to isomorphic data if and only if they are equal modulo swapping. The outer automorphism group of $\mathfrak{e}_{d^+,d^-}$ is trivial unless $d^+ = d^- = (d+1)/2$, in which case there is a nontrivial outer automorphism swapping H^+ and H^-.

These facts are well known (cf. [81, §1.8] or [73, §2.3]) and can be checked similarly as in the proof of [60, Proposition 2.1.1].

The even case.

Consider $d^+, d^- \in \mathbb{Z}_{\geq 0}$ both even and such that $d^+ + d^- = d$. We allow one of them to be zero. Denote $m^\pm := d^\pm/2$. Moreover consider $\delta^+, \delta^- \in \mathbb{R}^\times/\mathbb{R}^{\times,2} \cong \{\pm 1\}$, such that $\delta^+\delta^- = \delta$, the discriminant of (V, q). We assume none of $(d^\pm, \delta^\pm)$ is $(2, 1)$ or $(0, -1)$. Take $(V^\pm, q^\pm)$ to be quasi-split quadratic spaces of dimension $d^\pm$ and discriminant $\delta^\pm$ (cf. Definition 3.2.1). Note that we have $(V^+, q^+) \oplus (V^-, q^-) \cong (\underline{V}, \underline{q})$. Recall that all the three spaces $\underline{V}, V^+, V^-$ have been oriented.

Define s by

$$s_k := \begin{cases} 1, & m^- + 1 \leq k \leq d - m^- \\ -1, & \text{otherwise.} \end{cases}$$

On $\hat{H}^+ \times \hat{H}^-$, the map

$$\eta|_{\hat{H}^+ \times \hat{H}^-} : \hat{H}^+ \times \hat{H}^- = \mathrm{SO}_{d^+}(\mathbb{C}) \times \mathrm{SO}_{d^-}(\mathbb{C}) \to \hat{G} = \mathrm{SO}_d(\mathbb{C})$$

is given by the following map:

$$\mathbb{C}^{d^+} \times \mathbb{C}^{d^-} \xrightarrow{\sim} \mathbb{C}^d$$

$$(\hat{e}_k, 0) \mapsto \hat{e}_{k+m^-}$$

$$(0, \hat{e}_k) \mapsto \begin{cases} \hat{e}_k, \ k \leq m^- \\ \hat{e}_{k+d^+-1}, \ m^- + 1 \leq k \leq d^- \end{cases}.$$

We have already specified $\eta|_{\hat{H}}$. To specify η, it remains to define a map $\rho : W_{\mathbb{R}} \to \hat{G}$.

If $d^+ \neq 0$, we take $S \in \mathrm{GL}_d(\mathbb{C})$ to be the permutation matrix that switches $\hat{e}_{m^-}$ and $\hat{e}_{d-m^-+1}$, and switches $\hat{e}_m$ and $\hat{e}_{m+1}$, and leaves all the other $\hat{e}_i$'s fixed. (If $d^- = 0$, we will not need S anyway.) When $d^+ = 0$ take $S = \mathrm{id}$.

When $\delta^- = 1$, we take ρ to be trivial. Otherwise, we take $\rho(w) = 1$ for $w \in W_{\mathbb{C}}$ and $\rho(w) = S$ for $w \in W_{\mathbb{R}} - W_{\mathbb{C}}$.

Given a quadruple $(d^+, \delta^+, d^-, \delta^-)$ as above (disallowing $(2,1)$ and $(0,-1)$), we have constructed an (extended) elliptic endoscopic datum

$$\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H^+ \times H^-, s, \eta)$$

for G, well defined up to isomorphism. Every elliptic endoscopic datum of G, up to isomorphism, arises in this way, and two unequal quadruples $(d^+, \delta^+, d^-, \delta^-)$ and $(d_1^+, \delta_1^+, d_1^-, \delta_1^-)$ give rise to isomorphic data if and only if $(d^+, \delta^+) = (d_1^-, \delta_1^-)$ and $(d^-, \delta^-) = (d_1^+, \delta_1^+)$ (i.e. the two quadruples are equal up to swapping.) The outer automorphism group of $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$ is trivial when $d^+ d^- = 0$. When $d^+ = d^- = d/2$ and $\delta = 1$, the outer automorphism group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the first $\mathbb{Z}/2\mathbb{Z}$ acts by the simultaneous nontrivial outer automorphisms on H^+ and H^-, and the second $\mathbb{Z}/2\mathbb{Z}$ acts by swapping H^+ and H^-. In the rest case the outer automorphism group is $\mathbb{Z}/2\mathbb{Z}$, acting by the simultaneous nontrivial outer automorphisms on H^+ and H^-.

As in the odd case, these facts are well known. See the references given at the end of the discussion of the odd case.

Remark 4.3.12. We remark that for all the endoscopic data discussed above (in both the odd and the even case), the group $\bar{Z}$ discussed on p. 19 of [42] is trivial, as $Z(\hat{G}) = \{\pm 1\} \subset \hat{H}$. Hence the outer automorphism group of the datum is isomorphic to the automorphism group modulo $\hat{H}$. (cf. loc. cit.)

Remark 4.3.13. In the literature there are two different conventions of transfer factors (for a fixed convention of local class field theory, which we always choose to be the classical convention, cf. §0.7), which are related by $s \mapsto s^{-1}$. See [40, p. 184] and [35, §5.1] for more details. In our situation, s always satisfies $s^2 = 1$, so the two conventions are the same.

4.4 Cuspidality and transfer of elliptic tori

We keep the notation used in the previous subsection. Thus we have a quadratic space V over $\mathbb{R}$ of dimension d and discriminant δ, of arbitrary signature (p, q). Let $m := \lfloor d/2 \rfloor$. Let $G := \mathrm{SO}(V)$.

In the following we discuss a general way to obtain a pair (T_G, B), where T_G is an elliptic maximal torus of G (over $\mathbb{R}$) and B is a Borel subgroup of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$. This already appeared in our discussion in §4.1. Of course for T_G to exist G has to be cuspidal. When d is odd, G is cuspidal. When d is even, G is cuspidal if and only if $\delta = (-1)^{d/2}$. We see that the cuspidality condition is excluding exactly the case where p, q are both odd. In the following assume that at most one of p, q is odd.

Definition 4.4.1. An *elliptic decomposition (e.d.)* of V is an (ordered) m-tuple of oriented definite planes $V_1, \cdots, V_m$ in V, plus a line in the odd case, such that V is the orthogonal direct sum of them.

Definition 4.4.2. A *parametrized elliptic maximal torus* of G is an elliptic maximal torus T_G of G together with an isomorphism $\mathrm{U}(1)^m \xrightarrow{\sim} T_G$.

Definition 4.4.3. A *fundamental pair* in G is a Borel pair (T, B) in $G_{\mathbb{C}}$ with T defined over $\mathbb{R}$ and elliptic over $\mathbb{R}$.

Remark 4.4.4. Since any two elliptic maximal tori of G are conjugate under $G(\mathbb{R})$. the number of $G(\mathbb{R})$-orbits of fundamental pairs in G is equal to the cardinality of $\Omega_{\mathbb{C}}(G,T)/\Omega_{\mathbb{R}}(G,T)$, where T is an elliptic maximal torus of G. Namely, it is equal to the cardinality of a discrete series packet for $G(\mathbb{R})$.

Given an e.d. of V, we obtain a parametrized elliptic maximal torus by considering the embedding $\mathrm{U}(1)^m \hookrightarrow G$, where the i-th copy of $\mathrm{U}(1)$ acts by rotation on the oriented definite plane V_i. Now a parametrized elliptic maximal torus $\mathrm{U}(1)^m \xrightarrow{\sim} T_G$ gives rise to a Borel subgroup B of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$. Namely, we let B correspond to the standard order on the root datum on $(X^*(\mathrm{U}(1)^m), X_*(\mathrm{U}(1)^m) = (\mathbb{Z}^m, \mathbb{Z}^m)$, which is the standard root datum of type B or D. The composition of the above two constructions gives a map

$$\{\text{e.d. of } V\} \to \{\text{fundamental pairs in } G\}\,. \tag{4.4.1}$$

Now $G(\mathbb{R})$ naturally acts on both of the sets above, and the above map is equivariant.

In the odd case, two e.d.'s

$$(V_1, \cdots, V_m, l), \quad (V_1', \cdots, V_m', l')$$

are in the same $G(\mathbb{R})$-orbit if and only if V_i has the same definiteness as V_i' for each i. We see that the number of $G(\mathbb{R})$-orbits on both sides of (4.4.1) are equal to $\binom{d}{p}$ (which is equal to the size of a discrete series packet for $G(\mathbb{R})$).

In the even case, an e.d. gives rise to an orientation on V. Thus the set of e.d.'s is partitioned into two subsets according to the orientations on V. We call the subset corresponding to the orientation o_V in Definition 4.3.6 the set of *o_V-compatible e.d.'s.* Then two o_V-compatible e.d.'s

$$(V_1, \cdots, V_m), \quad (V_1', \cdots, V_m')$$

are in the same $G(\mathbb{R})$-orbit if and only if V_i has the same definiteness as V_i' for each i. We see that the number of $G(\mathbb{R})$-orbits in the set of o_V-compatible e.d.'s is $\binom{d}{p}$ (which is equal to the size of a discrete series packet for $G(\mathbb{R})$).

Lemma 4.4.5. *The map (4.4.1) is surjective. In the even case, it restricted to the set of o_V-compatible e.d.'s is also surjective.*

Proof. Let (T, B) be a fundamental pair in the image of (4.4.1). Let $(V_1, \cdots, V_m, l)$ (resp. $(V_1, \cdots, V_m)$) be a preimage in the odd (resp. even) case. In the even case assume the e.d. $(V_1, \cdots, V_m)$ is o_V-compatible. It suffices to prove that for any element ω in the complex Weyl group of (T, G), there exists an e.d. $\mathcal{D}_\omega$, which is o_V-compatible in the even case, such that the image of it under (4.4.1) is $(T, \omega B)$. In the odd case, the Weyl group is $\{\pm 1\}^m \rtimes \mathfrak{S}_m$. Write $\omega = ((a_1, \cdots, a_m), \sigma)$, with $a_i = \pm 1$ and $\sigma \in \mathfrak{S}_m$. We define $\mathcal{D}_\omega$ by first permuting the order of $V_1, \cdots, V_m$ according to σ and then changing the orientations on those V_i's such that $a_i = -1$. In the even case, we follow the same construction, only noting that $\mathcal{D}_\omega$ is still o_V-compatible because $(a_1, \cdots, a_m)$ lies in the subgroup $(\{\pm 1\}^m)'$ of $\{\pm 1\}^m$ defined by the condition that an even number of a_i's are -1. □

Corollary 4.4.6. *In the odd case the map (4.4.1) induces a bijection between $G(\mathbb{R})$-orbits. In the even case the map (4.4.1) restricted to the set of o_V-compatible e.d.'s induces a bijection between $G(\mathbb{R})$-orbits.*

Proof. This follows from Lemma 4.4.5 and cardinality counting in Remark 4.4.4 and above Lemma 4.4.5. □

Now recall from §4.3.2 that we have fixed an inner form of G, of the form $G^* = \mathrm{SO}(\underline{V})$, where $\underline{V}$ is a quasi-split quadratic space over $\mathbb{R}$ of the same dimension and discriminant as V. We have also chosen an isomorphism

$$\phi : V \otimes \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes \mathbb{C}$$

and denoted the induced inner twisting $G \xrightarrow{\sim} G^*$ by ψ. We have fixed a Borel pair $(\mathcal{B}, \mathcal{T})$ in $\hat{G}$ and have identified $\mathcal{T}$ with $\mathbb{G}_m^m$ in Definition 4.3.8. We have already assumed that G is cuspidal, so automatically G^* is also cuspidal. The above discussion of e.d. and parametrized elliptic maximal tori in G applies equally to $\underline{V}$ and G^*. Recall from Definition 4.3.3 and Definition 4.3.6 that in the even case we have fixed orientations $o_{\underline{V}}, o_V$ on $\underline{V}, V$ respectively.

Lemma 4.4.7. *Choose an e.d. of $\underline{V}$. In the even case assume it is $o_{\underline{V}}$-compatible. Let $\mathrm{U}(1)^m \xrightarrow{f} T_{G^*} \subset G^*$ be the resulting parametrized elliptic maximal torus of G^*, and let (T_{G^*}, B^*) be the resulting Borel pair in $G^*_{\mathbb{C}}$. Then the following diagram commutes:*

$$\begin{tikzcd} \widehat{\mathrm{U}(1)}^m \arrow[d, "can."] & \widehat{T_{G^*}} \arrow[l, "\hat{f}"'] \arrow[d, "B^*{,}\mathcal{B}"] \\ \mathbb{G}_m^m(\mathbb{C}) & \mathcal{T} \arrow[l] \end{tikzcd} .$$

Here the right vertical map is determined by $(B^*, \mathcal{B})$ and the left vertical map is the canonical identification.

Proof. Observe that in the odd and the even split case, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{U}(1)^m_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & T_{G^*,\mathbb{C}} \arrow[d] \\ \mathbb{G}^m_{m,\mathbb{C}} \arrow[r, "f_1"] & T'_{G^*,\mathbb{C}} \end{tikzcd} \tag{4.4.2}$$

where the left vertical arrow is the canonical isomorphism, the right vertical arrow is induced by $\mathrm{Int}(g)$ for some $g \in G(\mathbb{C})$, and the bottom horizontal arrow is the base change to $\mathbb{C}$ of a parametrized split maximal torus $f_1 : \mathbb{G}_m^m \hookrightarrow G^*$ considered in §4.3.7, associated to a chosen h.d. of $\underline{V}$, which is assumed to induce the fixed orientation on $\underline{V}$ in the even case. Thus it suffices to prove the commutativity of the following diagram:

$$\begin{tikzcd} \widehat{\mathbb{G}}^m \arrow[d, "\text{can.}"] & \widehat{T'_{G^*}} \arrow[l, "\hat{f}_1"'] \arrow[d, "\mathrm{Int}(g)(B^*),\mathcal{B}"] \\ \mathbb{G}_m^m(\mathbb{C}) & \mathcal{T} \arrow[l] \end{tikzcd}$$

By the discussion in §4.3.7, the above diagram commutes if and only if the based root datum given by $(T'_{G^*}, \mathrm{Int}(g)(B^*))$ is mapped under f_1^{-1} to the standard based root datum of type B on $(X^*(\mathbb{G}_m^m), X_*(\mathbb{G}_m^m)) = (\mathbb{Z}^m, \mathbb{Z}^m)$. By the commutativity of (4.4.2), the last condition is equivalent to the condition that the based root datum given by (T_{G^*}, B^*) is mapped under f^{-1} to the standard based root datum of type B on $(X^*(\mathrm{U}(1)^m), X_*(\mathrm{U}(1)^m)) = (\mathbb{Z}^m, \mathbb{Z}^m)$, but this is true by the definition of B^*.

In the even non-split case we need to replace "h.d." and "parametrized split maximal torus" by "n.h.d." and "parametrized near-split maximal torus", and the rest of the argument is the same. $\square$

Consider an elliptic endoscopic datum $\mathfrak{e}_{d^+,d^-}$ (resp. $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$) in the odd case (resp. even case). Recall that the endoscopic group in it is of the form $H = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$, where $V^\pm$ are quasi-split quadratic spaces over $\mathbb{R}$.

We assume that H is cuspidal, or equivalently, that both $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$ are cuspidal. Denote $m^\pm := \lfloor d^\pm/2 \rfloor$. Choose an e.d. of V^+ and and e.d. of V^-, and we get a parametrized elliptic maximal torus

$$f_H : \mathrm{U}(1)^{m^-} \times \mathrm{U}(1)^{m^+} \xrightarrow{\sim} T_{V^-} \times T_{V^+} \subset \mathrm{SO}(V^-) \times \mathrm{SO}(V^+). \tag{4.4.3}$$

To fix notation let us agree that whenever we identify $\mathrm{U}(1)^{m^-} \times \mathrm{U}(1)^{m^+}$ with $\mathrm{U}(1)^m$, it is the first m^- factors of $\mathrm{U}(1)^m$ that we identify with $\mathrm{U}(1)^{m^-}$.

Now we choose an e.d. of $\underline{V}$ and of V respectively, to get parametrized elliptic maximal tori

$$f^* : \mathrm{U}(1)^m \xrightarrow{\sim} T_{G^*} \subset G^* \tag{4.4.4}$$

$$f : \mathrm{U}(1)^m \xrightarrow{\sim} T_G \subset G. \tag{4.4.5}$$

The isomorphisms f_H, f^*, f in (4.4.3) (4.4.4) (4.4.5) with the common domain $\mathrm{U}(1)^m$ induce the following isomorphisms over $\mathbb{R}$:

$$T_{V^-} \times T_{V^+} \underset{\dagger}{\xrightarrow{\sim}} T_{G^*} \underset{\ddagger}{\xrightarrow{\sim}} T_G.$$

Lemma 4.4.8. *Assume either d is odd or d is even and the chosen e.d.'s of $\underline{V}$ and $V^{\pm}$ induce the fixed orientations on $\underline{V}$ and $V^{\pm}$. Then the isomorphism $\dagger : T_{V^-} \times T_{V^+} \xrightarrow{\sim} T_{G^*}$ is admissible, in the context of endoscopy.*

Proof. As in Definition 4.3.8 let $\mathcal{T} \cong \mathbb{G}_m^m$ be the diagonal maximal torus in the matrix group $\widehat{G}$. Analogously let $\mathcal{T}_{V^{\pm}} \cong \mathbb{G}_m^{m^{\pm}}$ be the diagonal maximal torus inside $\widehat{\mathrm{SO}(V^{\pm})}$. In view of Lemma 4.4.7 (applied to $\underline{V}$ and $V^{\pm}$, and for $V^{\pm}$ we consider the natural choice of Borel $\mathcal{B}_{V^{\pm}}$ in $\widehat{\mathrm{SO}(V^{\pm})}$ containing $\mathcal{T}_{V^{\pm}}$), we need only check that the map

$$\eta : \widehat{\mathrm{SO}(V^-)} \times \widehat{\mathrm{SO}(V^+)} \to \widehat{\mathrm{SO}(\underline{V})}$$

in the endoscopic datum induces the map $\mathcal{T}_{V^-} \times \mathcal{T}_{V^+} \to \mathcal{T}_{G^*}$ which corresponds to the natural map $\mathbb{G}_m^{m^-} \times \mathbb{G}_m^{m^+} \to \mathbb{G}_m^m$ on naturally identifying each of $\mathcal{T}_{V^-}, \mathcal{T}_{V^+}, \mathcal{T}_{G^*}$ as a product of $\mathbb{G}_m$'s. But the last statement follows from definition. $\square$

Remark 4.4.9. The above proof in fact shows that the map $\dagger$ is associated to the following four Borel pairs:

- The Borel pair in H associated to the chosen e.d.'s of V^+ and V^- via (4.4.1),
- The natural Borel pair $(\mathcal{T}_{\hat{H}}, \mathcal{B}_{\hat{H}}) = (\mathcal{T}_{V^-} \times \mathcal{T}_{V^+}, \mathcal{B}_{V^-} \times \mathcal{B}_{V^+})$ in $\hat{H}$.
- The natural Borel pair $(\mathcal{T}, \mathcal{B})$ in $\hat{G}$.
- The Borel pair in G^* associated to the chosen e.d. of $\underline{V}$ via (4.4.1).

Lemma 4.4.10. *Assume either d is odd or d is even and the chosen e.d.'s of V and $\underline{V}$ are o_V-compatible and $o_{\underline{V}}$-compatible. Then $\ddagger : T_{G^*} \xrightarrow{\sim} T_G$ is admissible, in the sense that it is equal to $\mathrm{Int}(g) \circ \psi^{-1}$, for some $g \in G(\mathbb{C})$.*

Proof. In this case the two ways of decomposing $V \otimes \mathbb{C}$ into an orthogonal direct sum of oriented planes (and perhaps a line), one induced by the chosen e.d. of V and the other induced by pulling back along ϕ of the chosen e.d. of $\underline{V}$, are related by an element of $G(\mathbb{C})$. Therefore there exists $g \in G(\mathbb{C})$ such that $g \circ \phi^{-1} : \underline{V} \otimes \mathbb{C} \to V \otimes \mathbb{C}$ sends the chosen e.d. of $\underline{V}$ to the chosen e.d. of V. But this shows that the embeddings (4.4.4) $\mathrm{U}(1)^m \hookrightarrow G^*$ and (4.4.5) $\mathrm{U}(1)^m \hookrightarrow G$ are related by $\mathrm{Int}(g) \circ \psi^{-1}$. $\square$

4.5 Transfer factors, the odd case

We follow the notation and setting of the previous subsection. We restrict to the odd case. Then G^* is adjoint and up to $G^*(\mathbb{R})$-conjugacy there is a unique equivalence class of

Whittaker data for G^*. Fix an elliptic endoscopic datum $(H = \mathrm{SO}(V^-) \times \mathrm{SO}(V^+), s, \eta)$ of G considered in §4.3.11.

The Whittaker normalization of the absolute geometric transfer factor between H and G^* was defined in [42, §5.3] (which can be specialized to the ordinary, i.e. untwisted, case), and the definition was later corrected in [35]. Note that by our convention on the local class field theory in §0.7, among the four transfer factors $\Delta, \Delta', \Delta_D, \Delta'_D$ at the end of [35, §5.1], we only consider Δ and Δ'. Moreover by Remark 4.3.13, we have $\Delta = \Delta'$. Thus according to our convention the *Whittaker normalized transfer factor* means the transfer factor Δ'_λ given in [35, (5.5.2)], and the discussion at the end of [35, §5.5] implies that $\Delta'_\lambda = \epsilon_L(V, \psi)\Delta'_0$, with $\Delta'_0 = \Delta_0$ (by Remark 4.3.13) where Δ_0 is defined on p. 248 of [53].

In the following we denote the Whittaker-normalized absolute geometric transfer factor between H and G^* by Δ^*_{Wh}. Also, having fixed $\psi : G \to G^*$ and $u(\rho) \in G^*(\mathbb{C})$ as in §4.3.2, , we can define the Whittaker normalization of the absolute geometric transfer factor between H and G as in [26]. We will recall this construction below.

We also consider the normalization $\Delta_{j,B}$ between H and G. See [40] for its definition. The goal of this subsection is to compare these two normalizations Δ_{Wh} and $\Delta_{j,B}$, under suitable simplifying assumptions.

For our application we are only interested in the case where V is of signature (p, q) with $p > q$. *In the following we assume this.*

4.5.1 Transfer factors between H and G^*

Definition 4.5.2. Let $(V_1, \cdots, V_m, l)$ be an e.d. of $\underline{V}$, where V_i's are oriented definite planes and l is a line. We say that it is *Whittaker*, if V_i is $(-1)^{m-i+1}\operatorname{sgn}(\delta)$-definite. Here δ is the discriminant of $\underline{V}$.

Remark 4.5.3. Let $(V_1, \cdots, V_m, l)$ be an arbitrary e.d. of $\underline{V}$. The line l is necessarily $\operatorname{sgn}(\delta)$-definite. Thus there are precisely $\lfloor m/2 \rfloor$ positive (resp. negative) definite planes among the

V_i's when $\delta > 0$ (resp. $\delta < 0$). Thus we can get a Whittaker e.d. by suitably re-ordering the V_i's.

Lemma 4.5.4. *Given a Whittaker e.d. $(V_1, \cdots, V_m, l)$ of $\underline{V}$, the resulting Borel pair in $G^*_{\mathbb{C}}$ via (4.4.1) has the property that every simple root is (imaginary) non-compact. In other words, in the terminology of [70], this Borel pair is a* fundamental pair of Whittaker type.

Proof. This follows easily from the definitions. Also see [72, §4.2.1]. We illustrate the argument with a special case, which actually contains all the computation needed in general. Suppose $m = 2$. Assume $\delta > 0$. The case $\delta < 0$ can be treated similarly. Then V_1 and l are positive definite and V_2 is negative definite. Let f_1, f_1' be an orthonormal basis of V_1 and f_2, f_2' be a negative-orthonormal basis[10] of V_2. Denote the complex conjugation by τ. Let

$$e_1 := f_1 + \sqrt{-1} f_1'$$

$$e_1' := \frac{1}{2} \tau e_1$$

$$e_2 := f_2 + \sqrt{-1} f_2'$$

$$e_2' := -\frac{1}{2} \tau e_2.$$

We also fix an orthonormal basis of l and still denote it by l. Let ϵ_1, ϵ_2 be the standard characters on $\mathrm{U}(1) \times \mathrm{U}(1)$ embedded in G^* via the e.d.. Then $\epsilon_i^\vee$ acts by weight 1 on e_i and weight -1 on e_i'. The simple roots are $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_2$. It suffices to check that for any root vector E_α of α and E_β of β, we have

$$[E_\alpha, \tau E_\alpha] = C H_\alpha \in \mathrm{Lie}\, G^*$$

$$[E_\beta, \tau E_\beta] = D H_\beta \in \mathrm{Lie}\, G^*$$

[10] i.e. $\underline{q}(f_2) = \underline{q}(f_2') = -1,\ [f_2, f_2'] = 0$.

for some $C, D \in \mathbb{R}_{>0}$. Here H_α is the coroot $\alpha^\vee$ viewed as an element of $\operatorname{Lie} G^*$, and similarly for H_β.

We take

$$E_\alpha : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$e_1 \mapsto 0$$

$$e_2 \mapsto e_1$$

$$e_1' \mapsto -e_2'$$

$$e_2' \mapsto 0$$

$$l \mapsto 0.$$

Then we have

$$\tau E_\alpha : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$e_2' \mapsto -e_1'$$

$$e_1' \mapsto 0$$

$$e_1 \mapsto e_2$$

$$e_2 \mapsto 0$$

$$l \mapsto 0.$$

We compute

$$[E_\alpha, \tau E_\alpha] : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$e_1 \mapsto e_1$$

$$e_2 \mapsto -e_2$$

$$e_1' \mapsto -e_1'$$

$$e_2' \mapsto e_2$$

$$l \mapsto 0,$$

which is equal to H_α, as desired.

We take

$$E_\beta : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$l \mapsto e_2$$

$$e_2' \mapsto -l$$

$$e_1, e_2, e_1' \mapsto 0.$$

Then we have

$$\tau E_\beta : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$l \mapsto -2e_2'$$

$$e_2 \mapsto 2l$$

$$e_1, e_1', e_2' \mapsto 0.$$

We compute

$$[E_\beta, \tau E_\beta] : \underline{V} \otimes \mathbb{C} \to \underline{V} \otimes \mathbb{C}$$

$$e_2 \mapsto 2e_2$$

$$e_2' \mapsto -2e_2'$$

$$e_1, e_1', l \mapsto 0,$$

which is equal to $2H_\beta$, as desired. $\square$

As before (Definition 4.3.8) we fix the natural Borel pair $(\mathcal{T}, \mathcal{B})$ in $\hat{G}$. We extend it to a pinning $\mathbf{spl}_{\hat{G}}$, which is automatically a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-pinning since the L-action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\hat{G}$ is trivial. Also we fix the natural Borel pair $(\mathcal{T}_{V^-} \times \mathcal{T}_{V^+}, \mathcal{B}_{V^-} \times \mathcal{B}_{V^+}) = (\mathcal{T}_{\hat{H}}, \mathcal{B}_{\hat{H}})$ in $\hat{H} = \widehat{\mathrm{SO}(V^-)} \times \widehat{\mathrm{SO}(V^+)}$. Now $(\mathcal{T}_{\hat{H}}, \mathcal{B}_{\hat{H}})$ is aligned with $(\mathcal{T}, \mathcal{B})$ in the sense that under the map

$$\eta : \mathcal{H} = {}^L H = \hat{H} \times \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to {}^L G = \hat{G} \times \mathrm{Gal}(\mathbb{C}/\mathbb{R}),$$

$\mathcal{T}_{\hat{H}}$ is identified with $\mathcal{T}$ and $\mathcal{B}_{\hat{H}}$ is equal to the pull-back of $\mathcal{B}$. We extend $(\mathcal{T}_{\hat{H}}, \mathcal{B}_{\hat{H}})$ to a pinning $\mathbf{spl}_{\hat{H}}$.

Here is the effect of having chosen $\mathbf{spl}_{\hat{G}}$ and $\mathbf{spl}_{\hat{H}}$ (cf. [69, §7.3, §8.1] , [68, §7]):

- Inside each equivalence class $\boldsymbol{\varphi}$ of elliptic Langlands parameters of G^* we can pick out a canonical $\mathcal{T}$-conjugacy class of parameters φ. We call elements φ in this $\mathcal{T}$-conjugacy class *almost-canonical representatives.*

- For each equivalence class $\boldsymbol{\varphi}$ of elliptic Langlands parameters of G^*, there is a canonical equivalence class $\boldsymbol{\varphi_H}$ of elliptic Langlands parameters of H, called *well-positioned*, that induce $\boldsymbol{\varphi}$ via $\eta : {}^L H \to {}^L G$.

- $\boldsymbol{\varphi_H}$ is characterized by $\boldsymbol{\varphi}$ as follows: One (hence any) almost-canonical representative φ_H for $\boldsymbol{\varphi_H}$ composed with $\eta : {}^L H \to {}^L G$ should be an almost-canonical representative φ for $\boldsymbol{\varphi}$.

We now choose an arbitrary $\boldsymbol{\varphi}$ and get $\boldsymbol{\varphi_H}$ as above. Choose an almost-canonical representative φ_H for $\boldsymbol{\varphi_H}$ and define φ to be the composition of φ_H and $\eta : {}^L H \to {}^L G$. Then φ is an almost-canonical representative for $\boldsymbol{\varphi}$. Thus the Borel pair in $\hat{G}$ (resp. $\hat{H}$) determined by φ (resp. φ_H) as on p. 182 of [40] is $(\mathcal{T}, \mathcal{B})$ (resp. $(\mathcal{T}_{\hat{H}}, \mathcal{B}_{\hat{H}})$).

Let π_0 be the unique generic member of $\Pi_{\boldsymbol{\varphi}}$ ([34], [77]). As in Shelstad [66] (cf. also [67, Theorem 3.6]), we have

$$\Delta^{*,\mathrm{Spec}}_{\mathrm{Wh}}(\boldsymbol{\varphi_H}, \pi_0) = 1. \tag{4.5.1}$$

Here $\Delta^{*,\mathrm{Spec}}_{\mathrm{Wh}}$ is the absolute spectral transfer factor between H and G^*, under the Whittaker normalization.

We now choose a Whittaker e.d. $\mathcal{D}^*$ of $\underline{V}$. Denote the resulting parametrized elliptic maximal torus by

$$\mathrm{U}(1)^m \xrightarrow[f^*]{\sim} T^* \subset G^*$$

and denote the resulting Borel pair by (T^*, B^*). Recall that these constructions are given in §4.4.

Choose an arbitrary e.d. $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ of $V^- \oplus V^+$. Denote the resulting parametrized elliptic maximal torus by

$$\mathrm{U}(1)^m \xrightarrow[f_H]{\sim} T_H \subset H$$

and denote the resulting Borel pair by (T_H, B_H). We also write $H^{\pm}$, $T_{H^{\pm}}$, $B_{H^{\pm}}$, and $f_{H^{\pm}}$, for the two components in each case.

When we want to emphasize the dependence of the objects defined above on $\mathcal{D}_H$ and $\mathcal{D}^*$, we will write $T^* = T_{\mathcal{D}^*}$, $f^* = f_{\mathcal{D}^*}$, $T_H = T_{\mathcal{D}_H}$, etc.

Define

$$j^* = j_{\mathcal{D}_H, \mathcal{D}^*} := f^* \circ f_H^{-1} : T_H \to T^*.$$

It is an isomorphism over $\mathbb{R}$. By Lemma 4.4.8 and Remark 4.4.9, the map j^* is admissible and (j^*, B_H, B^*) is aligned with φ_H in the sense of [40, p. 184].

If we write $\Delta^{\mathrm{Spec}}_{j^*,B^*}$ for the absolute spectral transfer factor normalized compatibly with the absolute geometric transfer factor Δ_{j^*,B^*} between H and G^*, then we have ([40, p. 185])

$$\Delta^{\mathrm{Spec}}_{j^*,B^*}(\boldsymbol{\varphi_H}, \pi(\varphi, \omega^{-1}B^*)) = \langle a_\omega, s\rangle,$$

for $\omega \in \Omega_{\mathbb{C}}(G^*, T^*)$. Here a_ω is defined in [40, §5]. For us it suffices to recall that

$$\langle a_1, s\rangle = 1.$$

Now by Vogan's classification theorem for generic representations [77, Theorem 6.2] and by Lemma 4.5.4, we know that $\pi_0 = \pi(\phi, B^*)$ and so it corresponds to $\omega = 1$. Hence

$$\Delta^{\mathrm{Spec}}_{j^*,B^*}(\boldsymbol{\varphi_H}, \pi_0) = 1. \tag{4.5.2}$$

Shelstad proves that the geometric and the spectral Whittaker normalizations are compatible ([68, Lemma 12.3]), so comparing 4.5.1 and 4.5.2 we get the following

Lemma 4.5.5. *Let $\mathcal{D}^*$ be a Whittaker e.d. of $\underline{V}$ and let $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ be an arbitrary e.d. for V^- and V^+. Let $j^* = j_{\mathcal{D}_H,\mathcal{D}^*}$ and let $B^* = B_{\mathcal{D}^*}$. Then*

$$\Delta^*_{\mathrm{Wh}} = \Delta_{j^*,B^*}.$$

□

4.5.6 Transfer factors between H and G

We keep the assumption that V is of signature (p, q) with $p > q$. Recall from Definition 4.3.4 that we have fixed $\phi : V \otimes \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes \mathbb{C}$.

Definition 4.5.7. Let $\mathcal{D}^* = (V_1, \cdots, V_m, l)$ be an e.d. of $\underline{V}$. We say that $\mathcal{D}^*$ is *suitable for* ϕ, if the following conditions hold:

1. For each i either $\phi^{-1}(V_i) \subset V$ or $\phi^{-1}(V_i) \subset V \otimes \sqrt{-1}$,

2. $\phi^{-1}(l) \subset V$, [11]

3. There exists i_0 such that $\phi^{-1}(V_i) \subset V \otimes \sqrt{-1}$ happens if and only if V_i is negative definite and $i \leq i_0$.

In this case, $\mathcal{D}^*$ determines a unique e.d. of V, of the form $\mathcal{D} = (V_1', \cdots, V_m', l')$, such that $V_i' = \phi^{-1}(V_i)$ or $\sqrt{-1}\phi^{-1}(V_i)$, and $l' = \phi^{-1}(l)$, and the orientation on V_i' is such that the map ϕ^{-1} or $\sqrt{-1}\phi^{-1}$ from V_i to V_i' preserves orientation. We write $\mathcal{D} = \phi^{-1}\mathcal{D}^*$.

As a consequence of the condition on ϕ assumed in Definition 4.3.4, there exists an e.d. of $\underline{V}$ that is suitable for ϕ. By re-ordering we can arrange it to be Whittaker.

Take a Whittaker e.d. $\mathcal{D}^* = (V_1, \cdots, V_m, l)$ of $\underline{V}$ suitable for ϕ, and let

$$\mathcal{D} = (V_1', \cdots, V_m', l') = \phi^{-1}\mathcal{D}^*$$

as in the above definition. Let $f^* : \mathrm{U}(1)^m \xrightarrow{\sim} T^* \subset G^*$ and $f : \mathrm{U}(1)^m \xrightarrow{\sim} T \subset G$ be the parametrized elliptic maximal tori associated to $\mathcal{D}^*$ and $\mathcal{D}$ respectively. Then the following diagram commutes:

$$\begin{tikzcd} \mathrm{U}(1)^m \arrow[r, "f"] \arrow[d, equal] & G \arrow[d, "\psi"] \\ \mathrm{U}(1)^m \arrow[r, "f^*"] & G^* \end{tikzcd}$$

namely, ψ sends the parametrized elliptic maximal torus (T, f) in G to (T^*, f^*) in G^*.

In particular, $u(\rho)$ defined in §4.3.2 lies in $T^*(\mathbb{C})$. More precisely, for $\rho = \tau$ the complex conjugation, $u(\tau)$ acts as -1 on those V_i such that $\phi^{-1}V_i \subset V \otimes \sqrt{-1}$, and acts as 1 on l and on those V_i such that $\phi^{-1}V_i \subset V$. This implies that in fact $u(\rho) \in T^*(\mathbb{R})$.

[11] Given condition (1), it is a priori impossible to have $\phi^{-1}(l) \subset V \otimes \sqrt{-1}$, since $\underline{V}$ and V have the same discriminant.

Take any $\mathcal{D}_H$ and $j^* : T_H \to T^*$ as in Lemma 4.5.5 and let $B^* := B_{\mathcal{D}^*}, B_H := B_{\mathcal{D}_H}$. Take a test element $\delta^* \in T^*(\mathbb{R})$ that is strongly regular in G^*, and define $\gamma := (j^*)^{-1}(\delta^*) \in T_H(\mathbb{R})$ and $\delta := \psi^{-1}(\delta^*) \in T(\mathbb{R})$.

Kaletha [26, §2.2] proves that there is a normalization Δ_{Wh} of the absolute geometric transfer factor between H and G characterized by the following formula:

$$\Delta_{\mathrm{Wh}}(\gamma, \delta) = \Delta^*_{\mathrm{Wh}}(\gamma, \delta^*)\langle \mathrm{inv}(\delta, \delta^*), s_{\gamma,\delta^*}\rangle^{-1} \tag{4.5.3}$$

where:

- Δ^*_{Wh} is the Whittaker normalization of the transfer factor between H and G^*.
- $\mathrm{inv}(\delta, \delta^*)$ is the image of the cocycle $(\rho \mapsto u(\rho))$ under the Tate-Nakayama isomorphism $\mathbf{H}^1(\mathbb{R}, T^*) \xrightarrow{\sim} \mathbf{H}^{-1}(\Gamma_\infty, X_*(T^*)) = X_*(T^*)_{\Gamma_\infty}$. (Since T^* is a product of U(1), the norm map on $X_*(T^*)$ is zero.)
- s_{γ,δ^*} is the image of $s \in Z(\hat{H})$ in the endoscopic datum under the maps

  $$Z(\hat{H}) \hookrightarrow \widehat{T_H} \xrightarrow{\hat{j}} \widehat{T^*} = (\widehat{T^*})^{\Gamma_\infty},$$

  where the first map is given by any admissible isomorphism between $\widehat{T_H}$ and $\mathcal{T}_{\hat{H}}$. [12]

Definition 4.5.8. We call Δ_{Wh} the *Whittaker normalization of the absolute geometric transfer factor between H and G.*

Lemma 4.5.9. *Keep the notation as in (4.5.3). We have*

$$\langle \mathrm{inv}(\delta, \delta^*), s_{\gamma,\delta^*}\rangle = (-1)^{k(\mathcal{D}^*)},$$

[12] Any two such isomorphisms differ by an element in $\Omega(\mathcal{T}_{\hat{H}}, \hat{H})$, so they restrict to the same map on $Z(\hat{H}) \subset \mathcal{T}_{\hat{H}}$

where

$$k(\mathcal{D}^*) = \#\left\{1 \le i \le m^- | \phi^{-1} V_i \subset V \otimes \sqrt{-1}\right\}.$$

Proof. By [26, Lemma 2.3.3], the element $\mathrm{inv}(\delta, \delta^*) \in X_*(T^*)_{\Gamma_\infty}$ is equal to image of any element $\lambda \in X_*(T^*)$ such that $\lambda(-1) = u(\tau)$, where τ is the complex conjugation. The lemma then follows easily from the description of $u(\tau)$ above (4.5.3). □

Define $j : T_H \xrightarrow{\sim} T$ to be the composition $\psi^{-1} \circ j^*$. Then j is an admissible isomorphism. We have the Borel subgroup $B_{\mathcal{D}}$ of $G_{\mathbb{C}}$ containing $T_{\mathbb{C}}$ determined by $\mathcal{D}$ via (4.4.1). Our goal is to compare $\Delta_{j,B_{\mathcal{D}}}$ and Δ_{Wh} as transfer factors between H and G.

Lemma 4.5.10. *Suppose the signature of V is (p, q) with $p > q$.*[13] *Then*

$$\Delta_{j^*,B^*}(\gamma, \delta^*) = (-1)^{(m-p)(m+1-p)/2} \Delta_{j,B_{\mathcal{D}}}(\gamma, \delta).$$

Proof. By the formula for $\Delta_{j,B}$ on p. 184 of [40], we have

$$\Delta_{j^*,B^*}(\gamma, \delta^*) = (-1)^{q(G^*)+q(H)} \chi_{G^*,H}(\delta^*) \Delta_{B^*}((\delta^*)^{-1}) \Delta_{B_H}(\gamma^{-1})^{-1}$$

$$\Delta_{j,B_{\mathcal{D}}}(\gamma, \delta) = (-1)^{q(G)+q(H)} \chi_{G,H}(\delta) \Delta_{B_{\mathcal{D}}}(\delta^{-1}) \Delta_{B_H}(\gamma^{-1})^{-1}.$$

We have $q(\mathrm{SO}(a, b)) = ab/2$, so

$$q(G^*) - q(G) = \frac{(m+1)m}{2} - \frac{p(2m+1-p)}{2} = \frac{(m-p)(m+1-p)}{2}.$$

It follows from the fact that ψ sends the Borel pair $(T, B_{\mathcal{D}})$ to (T^*, B^*) that

$$\Delta_{B^*}((\delta^*)^{-1}) = \Delta_{B_{\mathcal{D}}}(\delta^{-1}).$$

[13] For this lemma it is not necessary to assume $p > q$.

It remains to show that

$$\chi_{G^*,H}(\delta^*) = \chi_{G,H}(\delta).$$

By definition it suffices to check that the following diagram commutes up to $\hat{G}$-conjugation:

$$\begin{tikzcd} {}^L T \arrow[r, "\eta_{B_{\mathcal{D}}}"] \arrow[d] & {}^L G \arrow[d, equal] \\ {}^L(T^*) \arrow[r, "\eta_{B^*}"] & {}^L G \end{tikzcd}$$

where the left vertical arrow is induced by $\psi : T \xrightarrow{\sim} T^*$ ($\mathbb{R}$-isomorphism). But this is true by the characterization (a)(b) on p. 183 of [40], in view of the fact, again, that ψ sends $(T, B_{\mathcal{D}})$ to (T^*, B^*). $\square$

For our application it suffices to make the simplifying assumption in the following proposition.

Proposition 4.5.11. *Assume that the signature (p, q) of V satisfies that $p > q$, ($p+q$ odd), and assume*

$$\begin{cases} q \text{ even} \\ q/2 \leq \lceil m^+/2 \rceil \end{cases} \quad \text{or} \quad \begin{cases} q \text{ odd} \\ (q-1)/2 \leq \lfloor m^+/2 \rfloor \end{cases}$$

Then

$$\Delta_{j,B_{\mathcal{D}}} = \begin{cases} (-1)^{\lceil \frac{m}{2} \rceil + \lceil \frac{m^+}{2} \rceil + \lceil \frac{m-p}{2} \rceil} \Delta_{\mathrm{Wh}}, & \delta > 0 \\ (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m^+}{2} \rfloor + \lceil \frac{m-p}{2} \rceil} \Delta_{\mathrm{Wh}}, & \delta < 0. \end{cases}$$

In particular, we have

$$\Delta_{j,B_{\mathcal{D}}} = \begin{cases} (-1)^{1+\lceil \frac{m^+}{2} \rceil} \Delta_{\mathrm{Wh}}, & q = 2 \\ (-1)^{\lceil \frac{m^+}{2} \rceil} \Delta_{\mathrm{Wh}}, & q = 0 \\ (-1)^{\lfloor \frac{m^+}{2} \rfloor} \Delta_{\mathrm{Wh}}, & q = 1. \end{cases}$$

Proof. Let γ, δ, δ^* be as above. By Lemma 4.5.5, (4.5.3), Lemma 4.5.9, Lemma 4.5.10, we have

$$\Delta_{j,B_{\mathcal{D}}}(\gamma, \delta) = (-1)^{k(\mathcal{D}^*)+(m-p)(m+1-p)/2} \Delta_{\mathrm{Wh}}(\gamma, \delta).$$

Now note that among the m^+ planes $V_{m^-+1}, V_{m^-+2}, \cdots, V_m$, there are precisely $\lceil m^+/2 \rceil$ (resp. $\lfloor m^+/2 \rfloor$) that are negative definite when $\delta > 0$ (resp. $\delta < 0$). Under our assumption on (p, q), there does not exist $i \leq m^-$ such that V_i is negative definite and $\phi^{-1}(V_i) \subset V$, because otherwise ϕ^{-1} would send this V_i, together with all the negative definite planes among $V_{m^-+1} \cdots, V_m$, to negative definite planes in V, and we would have $\lfloor q/2 \rfloor$ strictly larger than the number of negative definite planes among $V_{m^-+1} \cdots, V_m$, a contradiction. Hence

$$\begin{aligned} k(\mathcal{D}^*) &= \# \left\{ 1 \leq i \leq m^- | \phi^{-1} V_i \subset V \otimes \sqrt{-1} \right\} \\ &= \# \left\{ 1 \leq i \leq m^- | V_i \text{ is negative definite} \right\}. \end{aligned}$$

When $\delta > 0$,

$$k(\mathcal{D}^*) = \begin{cases} \lceil m^-/2 \rceil, & m \text{ odd} \\ \lfloor m^-/2 \rfloor, & m \text{ even} \end{cases} \equiv \lceil m/2 \rceil + \lceil m^+/2 \rceil \mod 2.$$

When $\delta < 0$,

$$k(\mathcal{D}^*) = \begin{cases} \lceil m^-/2 \rceil, & m \text{ even} \\ \lfloor m^-/2 \rfloor, & m \text{ odd} \end{cases} \equiv \lfloor m/2 \rfloor + \lfloor m^+/2 \rfloor \mod 2.$$

We check for each parity of m that

$$\frac{(m-p)(m+1-p)}{2} \equiv \lceil \frac{m-p}{2} \rceil \mod 2$$

$\square$

Definition 4.5.12. Let $\mathcal{D}_1 = (V_1'', \cdots, V_m'', l'')$ be an arbitrary e.d. of V. We say that $\mathcal{D}_1$ is *convenient*, if there exists i_0 such that

$$\{i | V_i'' \text{ negative definite}\} = \left\{ i | i > i_0 \text{ and } (-1)^{i+m} = \operatorname{sgn} \delta \right\}.$$

In particular when $q = 2$, $\mathcal{D}_1$ is convenient if V_m'' is negative definite. When $q = 1$ or 0, any e.d. of V is convenient.

Remark 4.5.13. $\mathcal{D}_1$ is convenient if and only if it is in the $G(\mathbb{R})$-orbit of $\mathcal{D} = \phi^{-1}(\mathcal{D}^*)$ considered above.

Corollary 4.5.14. *Keep the assumption on (p, q) as in Proposition 4.5.11. Let $\mathcal{D}_1$ be an arbitrary convenient e.d. of V. Let $f_1 : \mathrm{U}(1)^m \xrightarrow{\sim} T_1 \subset G$ be the associated parametrized elliptic maximal torus and let (T_1, B_1) be the associated fundamental pair. Let $\mathcal{D}_H$ be an arbitrary e.d. of $V^- \oplus V^+$ and let $f_H : \mathrm{U}(1)^m \xrightarrow{\sim} T_H \subset H$ be the associated parametrized elliptic maximal torus. Let $j_1 := f_1 \circ f_H^{-1} : T_H \xrightarrow{\sim} T_1$. Then j_1 is admissible and we have*

$$\Delta_{j_1, B_1} = \begin{cases} (-1)^{\lceil \frac{m}{2} \rceil + \lceil \frac{m^+}{2} \rceil + \lceil \frac{m-p}{2} \rceil} \Delta_{\mathrm{Wh}}, & \delta > 0 \\ (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m^+}{2} \rfloor + \lceil \frac{m-p}{2} \rceil} \Delta_{\mathrm{Wh}}, & \delta < 0 \end{cases}$$

as absolute geometric transfer factors between H and G. In particular, we have

$$\Delta_{j_1,B_1} = \begin{cases} (-1)^{1+\lceil \frac{m^+}{2} \rceil}\Delta_{\mathrm{Wh}},\ q=2 \\ (-1)^{\lceil \frac{m^+}{2} \rceil}\Delta_{\mathrm{Wh}},\ q=0 \\ (-1)^{\lfloor \frac{m^+}{2} \rfloor}\Delta_{\mathrm{Wh}},\ q=1. \end{cases}$$

Proof. By Remark 4.5.13, we know that Δ_{j_1,B_1} is equal to $\Delta_{j,B_{\mathcal{D}}}$ considered in Proposition 4.5.11. □

4.6 Transfer factors, the even case with $d/2$ odd

We follow the notation and setting of §4.4. We assume d is even and $m = d/2$ is odd. Recall that we have assumed the cuspidality condition, namely $\delta = (-1)^{d/2} = -1$. In particular G^* is not split. In this case G^* has a unique equivalence of Whittaker data, because the map $G^*(\mathbb{R}) \to (G^*)^{\mathrm{ad}}(\mathbb{R})$ is surjective, which can be seen by noting that $\ker \mathbf{H}^1(\mathbb{R}, Z_{G^*}) \to \mathbf{H}^1(\mathbb{R}, G^*)$ is trivial. Moreover, as we have been doing in the above, we do not just view G as an abstract algebraic group, but we realizes G as a pure inner form of $G^* = \mathrm{SO}(\underline{V})$, via the maps

$$\phi : (V, q) \otimes \mathbb{C} \xrightarrow{\sim} (\underline{V}, \underline{q}) \otimes \mathbb{C}$$

and

$$\psi : G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}$$

discussed in §4.3.2.

All the discussions in §4.5 carry over analogously, as follows.

4.6.1 Transfer factors between H and G^*

Definition 4.6.2. Let $(V_1, \cdots, V_m)$ be an $o_{\underline{V}}$-compatible e.d. of $\underline{V}$. We say that it is *Whittaker* if V_i is $(-1)^{i+1}$-definite.

Remark 4.6.3. Let $(V_1, \cdots, V_m)$ be an arbitrary e.d. of $\underline{V}$. Then there are precisely $\lceil m/2 \rceil$ positive definite planes among the V_i's. Thus we can get a Whittaker e.d. by suitably re-ordering the V_i's and changing the orientations on some of the V_i's.

Lemma 4.6.4. *Given a Whittaker e.d. $(V_1, \cdots, V_m, l)$ of $\underline{V}$, the resulting Borel pair in $G^*_{\mathbb{C}}$ via (4.4.1) has the property that every simple root is (imaginary) non-compact.*

Proof. This can be checked by easy computation as Lemma 4.5.4. See [72, §4.2.1]. □

Fix an elliptic endoscopic datum $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H = \mathrm{SO}(V^-) \times \mathrm{SO}(V^+), s, \eta)$ of G considered in 4.3.11. Suppose H is cuspidal. Thus $\delta^\pm = (-1)^{d^\pm/2}$.

Choose a Whittaker e.d. $\mathcal{D}^*$ of $\underline{V}$, and an arbitrary e.d. $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ of V^- and V^+, such that they induce the given orientations on V^- and V^+. By Lemma 4.4.8, we get an admissible isomorphism $j_{\mathcal{D}_H,\mathcal{D}^*} : T_{\mathcal{D}_H} \xrightarrow{\sim} T_{\mathcal{D}^*}$. Let $B_{\mathcal{D}^*}$ be the Borel containing $T_{\mathcal{D}^*,\mathbb{C}}$ determined by $\mathcal{D}^*$ via (4.4.1).

Lemma 4.6.5. $\Delta^*_{\mathrm{Wh}} = \Delta_{j_{\mathcal{D}_H,\mathcal{D}^*}, B_{\mathcal{D}^*}}$*, where Δ^*_{Wh} is the Whittaker normalized absolute geometric transfer factor (with respect to the unique equivalence class of Whittaker data for G^*), for which we follow the convention discussed in the beginning of §4.5.*

Proof. The proof is the same as Lemma 4.5.5. □

4.6.6 Transfer factors between H and G.

Assume the signature of V is (p, q) with $p > q$. The cuspidality condition forces p and q to be even. Recall from Definition 4.3.4 that we have fixed $\phi : V \otimes \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes \mathbb{C}$.

Definition 4.6.7. Let $\mathcal{D}^* = (V_1, \cdots, V_m)$ be an $o_{\underline{V}}$-compatible e.d. of $\underline{V}$. We say that $\mathcal{D}^*$ is *suitable for* ϕ, if the following conditions hold:

1. For each i either $\phi^{-1}(V_i) \subset V$ or $\phi^{-1}(V_i) \subset V \otimes \sqrt{-1}$.

2. There exists i_0 such that $\phi^{-1}(V_i) \subset V \otimes \sqrt{-1}$ happens if and only if V_i is negative definite and $i \leq i_0$.

In this case $\mathcal{D}^*$ determines an e.d. of V analogously as Definition 4.5.7, denoted by $\phi^{-1}\mathcal{D}^*$. We note that $\phi^{-1}\mathcal{D}^*$ is o_V-compatible.

Definition 4.6.8. As in the odd case (cf. (4.5.3) and Definition 4.5.8), having fixed ϕ we obtain the notion of the *Whittaker normalization of the absolute geometric transfer factor between* H *and* G. We denote this by Δ_{Wh}.

As a consequence of the condition on ϕ assumed in Definition 4.3.4, there exists an e.d. of $\underline{V}$ that is suitable for ϕ. By re-ordering we can arrange it to be Whittaker. We thus let $\mathcal{D}^*$ be a Whittaker e.d. of $\underline{V}$ that is suitable for ϕ. Let $\mathcal{D} := \phi^{-1}\mathcal{D}^*$. Take an arbitrary e.d. $\mathcal{D}_H$ for $V^+ \times V^-$ giving rise to the chosen orientations on $V^\pm$. Define $j = \psi^{-1} j_{\mathcal{D}_H, \mathcal{D}^*}$. By using analogous statements as Lemma 4.5.9 and Lemma 4.5.10, we get

Lemma 4.6.9. $\Delta_{j,B_{\mathcal{D}}} = (-1)^{q(G)+q(G^*)+k(\mathcal{D}^*)}\Delta_{\mathrm{Wh}} = (-1)^{k(\mathcal{D}^*)}\Delta_{\mathrm{Wh}}$, *(noting that* $q(G)$ *and* $q(G^*)$ *are even), where*

$$k(\mathcal{D}^*) := \#\left\{1 \leq i \leq m^- | \phi^{-1}V_i \subset V \otimes \sqrt{-1}\right\}.$$

□

Proposition 4.6.10. *When* $q/2 \leq \lfloor m^+/2 \rfloor$, *we have* $\Delta_{j,B_{\mathcal{D}}} = (-1)^{\lfloor m^-/2 \rfloor}\Delta_{\mathrm{Wh}}$. *When* $m^+ = 1$ *and* $q = 2$, *we have* $\Delta_{j,B_{\mathcal{D}}} = (-1)^{\frac{m^-}{2}-1}\Delta_{\mathrm{Wh}}$.

Proof. Among the m^+ planes $V_{m^-+1}, \cdots, V_m$, precisely $\lfloor m^+/2 \rfloor$ are negative definite. Hence in the first cases there does not exist $i \leq m^-$ such that V_i is negative definite and $\phi^{-1}(V_i) \subset V$. Thus

$$k(\mathcal{D}^*) = \# \left\{ 1 \leq i \leq m^- | V_i \text{ is negative definite} \right\} = \lfloor m^-/2 \rfloor.$$

Now assume $m^+ = 1, q = 2$. Then

$$k(\mathcal{D}^*) = \# \left\{ 1 \leq i \leq m^- | V_i \text{ is negative definite} \right\} - 1 = \lfloor m^-/2 \rfloor - 1 = \frac{m^-}{2} - 1.$$

The proposition now follows from Lemma 4.6.9. $\square$

Definition 4.6.11. Let $\mathcal{D}_1 = (V_1'', \cdots, V_m'')$ be an arbitrary e.d. of V. We say that $\mathcal{D}_1$ is *convenient*, if it is o_V-compatible, and such that there exists i_0 satisfying

$$\{ i | V_i'' \text{ negative definite} \} = \{ i | i > i_0 \text{ and } i \equiv 0 \mod 2 \}.$$

Remark 4.6.12. $\mathcal{D}_1$ is convenient if and only if it is in the $G(\mathbb{R})$-orbit of $\mathcal{D} = \phi^{-1}(\mathcal{D}^*)$ considered above.

Corollary 4.6.13. *Let $\mathcal{D}_1$ be an arbitrary convenient e.d. of V. Let $f_1 : \mathrm{U}(1)^m \xrightarrow{\sim} T_1 \subset G$ be the associated parametrized elliptic maximal torus and let (T_1, B_1) be the associated fundamental pair. Let $\mathcal{D}_H$ be an arbitrary e.d. of $V^- \oplus V^+$ giving rise to the fixed orientations on $V^\pm$ and let $f_H : \mathrm{U}(1)^m \xrightarrow{\sim} T_H \subset H$ be the associated parametrized elliptic maximal torus. Let $j_1 := f_1 \circ f_H^{-1} : T_H \xrightarrow{\sim} T_1$. Then j_1 is admissible and we have*

$$\Delta_{j_1, B_1} \Delta_{\mathrm{Wh}}^{-1} = \begin{cases} (-1)^{\lfloor m^-/2 \rfloor}, & q = 0 \\ (-1)^{\lfloor m^-/2 \rfloor}, & q = 2,\ m^+ \geq 2 \\ (-1)^{\lfloor m^-/2 \rfloor + 1}, & q = 2,\ m^+ = 1. \end{cases}$$

4.7 Transfer factors, the even case with $m/2$ even

We follow the notation and setting of §4.4. We assume d is even and $m = d/2$ is even. By the cuspidality condition we have $\delta = (-1)^{d/2} = 1$. In particular G^* is split. We would like to establish results analogous to those in §4.5 and §4.6. The new feature and difficulty in the current case is that there are in total two equivalent classes of Whittaker data of G^*. To ease the language, in the following we will use the phrase "Whittaker datum" to mean an equivalence class of Whittaker data. Having realized G^* as $\mathrm{SO}(\underline{V}, \underline{q})$ (namely remembering $(\underline{V}, \underline{q})$ as part of the data), we are able to distinguish (in a somewhat abstract way) the two Whittaker data and call them *type I* and *type II*. Roughly speaking, the reason that there are two Whittaker data is because $(\underline{V}, \underline{q})$ is in isometry with $(\underline{V}, -\underline{q})$. We shall establish results analogous to §4.5 and §4.6 only for the type I Whittaker datum. One extra task we need to take for the application is to study how the notion of being type I passes to cuspidal Levi subgroups. This we shall carry out in the end of this subsection. In the following many remarks and arguments in §4.6 carry over, and we do not repeat them.

4.7.1 Transfer factors between H and G^*

Definition 4.7.2. Let $(V_1, \cdots, V_m)$ be an $o_{\underline{V}}$-compatible e.d. of $\underline{V}$. We say that it is *type I Whittaker* if V_i is $(-1)^{i+1}$-definite. We say that it is *type II Whittaker* if V_i is $(-1)^i$-definite.

Let (T_I, B_I) (resp. (T_{II}, B_{II})) be the fundamental pair associated to a type I (resp. type II) Whittaker e.d.. Then they both satisfy the condition that every simple root is non-compact. As in [72, §4.2.1], (T_I, B_I) and (T_{II}, B_{II}) correspond to two Whittaker data $\mathfrak{w}_I$ and $\mathfrak{w}_{II}$ of G^* respectively, characterized by the condition that in any discrete series packet for $G^*(\mathbb{R})$ the element corresponding to (T_I, B_I) (resp. (T_{II}, B_{II})) is generic with respect to $\mathfrak{w}_I$ (resp. $\mathfrak{w}_{II}$). The Whittaker data $\mathfrak{w}_I$ and $\mathfrak{w}_{II}$ are inequivalent and hence exhaust the equivalent classes of Whittaker data for G^*. See loc. cit. for more details.

Fix an elliptic endoscopic datum $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H = \mathrm{SO}(V^-) \times \mathrm{SO}(V^+), s, \eta)$ of G considered in 4.3.11. Suppose H is cuspidal. As usual let $m^\pm$ be the absolute ranks of $H^\pm$. Thus $\delta^\pm = (-1)^{m^\pm}$.

Definition 4.7.3. We denote by Δ^*_{Wh} the Whittaker normalized absolute geometric transfer factor between H and G^* with respect to $\mathfrak{w}_I$. Denote by $\tilde{\Delta}^*_{\mathrm{Wh}}$ the analogous object with respect to $\mathfrak{w}_{II}$.

Choose a type I Whittaker e.d. $\mathcal{D}^*$ of $\underline{V}$, and an arbitrary e.d. $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ of V^- and V^+, such that they induce the given orientations on V^- and V^+. By Lemma 4.4.8, we get an admissible isomorphism $j_{\mathcal{D}_H,\mathcal{D}^*} : T_{\mathcal{D}_H} \xrightarrow{\sim} T_{\mathcal{D}^*}$. Let $B_{\mathcal{D}^*}$ be the Borel containing $T_{\mathcal{D}^*,\mathbb{C}}$ determined by $\mathcal{D}^*$ via (4.4.1).

Lemma 4.7.4. *We have*

- $\Delta^*_{\mathrm{Wh}} = \Delta_{j_{\mathcal{D}_H,\mathcal{D}^*},B_{\mathcal{D}^*}}$.
- $\tilde{\Delta}^*_{\mathrm{Wh}} = (-1)^{m^-}\Delta_{j_{\mathcal{D}_H,\mathcal{D}^*},B_{\mathcal{D}^*}}$. *Here as usual we denote by $m^\pm$ the absolute ranks of $H^\pm$. In particular $\tilde{\Delta}^*_{\mathrm{Wh}} = \Delta^*_{\mathrm{Wh}}$.*

Proof. The proof of the first statement is the same as Lemma 4.5.5. For the second statement, we also use similar arguments as those for Lemma 4.5.5, and we are reduced to checking that

$$\langle a_\omega, s\rangle = (-1)^{m+-},$$

where ω is the element of the complex Weyl group of $T_{\mathcal{D}^*}$ relating the two Borels corresponding to a type I and a type II Whittaker e.d.. We easily see that

$$\omega = (12)(34)\cdots(m-1,m) \in \mathfrak{S}_m \subset \Omega_{\mathbb{C}}(G^*, T_{\mathcal{D}^*})$$

and the class $a_\omega \in \mathbf{H}^1(\mathbb{R}, T_{\mathcal{D}^*})$ is represented by the cocycle sending the complex conjugation τ to $-1 \in T_{\mathcal{D}^*}(\mathbb{R})$. It easily follows that $\langle a_\omega, s\rangle = (-1)^{m^-}$.

Alternatively, the second statement also follows from the observation that if one changes the sign of $\eta \in \mathbb{R}^\times/\mathbb{R}^{\times,2}$ in Waldspurger's explicit formula in [81, §1.10] specialized to G^* and H, the resulting sign change is $(-1)^{m^-}$ (cf. (4.7.9) below). This proves our statement because Waldspurger's formula is for the Langlands-Shelstad normalization Δ_0 (for a quasi-split group with a fixed Galois invariant pinning),[14] and changing the pinning of G^* has the same effect on the Langlands-Shelstad normalization Δ_0 as the effect of changing the Whittaker datum on the Whittaker normalization. $\square$

4.7.5 Transfer factors between H and G.

Assume the signature of V is (p, q) with $p > q$. The cuspidality condition forces p and q to be even. Recall from Definition 4.3.4 that we have fixed $\phi : V \otimes \mathbb{C} \xrightarrow{\sim} \underline{V} \otimes \mathbb{C}$.

Definition 4.7.6. Let $\mathcal{D}^* = (V_1, \cdots, V_m)$ be an $o_{\underline{V}}$-compatible e.d. of $\underline{V}$. We define the notion that $\mathcal{D}^*$ is *suitable for* ϕ, and define $\phi^{-1}\mathcal{D}^*$ for such $\mathcal{D}^*$, in exactly the same way as Definition 4.6.7.

Definition 4.7.7. As in the odd case (cf. (4.5.3) and Definition 4.5.8), having fixed ϕ and with the choice $\mathfrak{w}_I$ of the Whittaker datum, we obtain the notion of the *type I Whittaker normalization of the absolute geometric transfer factor between H and G.* We denote this by Δ_{Wh}.

Remark 4.7.8. Analogously we also have the type II Whittaker normalization between H and G. By definition and by the second statement in Lemma 4.7.4, it is equal to $(-1)^{m^-}\Delta_{\mathrm{Wh}}$.

Similarly as in §4.6, we let $\mathcal{D}^*$ be a type I Whittaker e.d. of $\underline{V}$ that is suitable for ϕ. Let $\mathcal{D} := \phi^{-1}\mathcal{D}^*$. Take an arbitrary e.d. $\mathcal{D}_H$ for $V^+ \times V^-$ giving rise to the chosen orientations

[14] Waldspurger omits the factor Δ_{IV}, which is a positive real number and hence harmless for this discussion.

on $V^{\pm}$. Define $j = \psi^{-1} j_{\mathcal{D}_H,\mathcal{D}^*}$. By totally similar arguments as loc. cit. we have

Proposition 4.7.9. *Assume $q/2 \leq \lceil m^+/2 \rceil$. Then $\Delta_{j,B_{\mathcal{D}}} = (-1)^{\lfloor m^-/2 \rfloor} \Delta_{\mathrm{Wh}}$.* $\square$

Definition 4.7.10. Let $\mathcal{D}_1 = (V_1'', \cdots, V_m'')$ be an arbitrary e.d. of V. We say that $\mathcal{D}_1$ is *type I convenient*, or simply *convenient*, if it is o_V-compatible, and such that there exists i_0 satisfying

$$\{i | V_i'' \text{ negative definite}\} = \{i | i > i_0 \text{ and } i \equiv 0 \mod 2\}.$$

Remark 4.7.11. $\mathcal{D}_1$ is convenient if and only if it is in the $G(\mathbb{R})$-orbit of $\mathcal{D} = \phi^{-1}(\mathcal{D}^*)$ considered above.

Corollary 4.7.12. *Let $\mathcal{D}_1$ be an arbitrary convenient e.d. of V. Let $f_1 : \mathrm{U}(1)^m \xrightarrow{\sim} T_1 \subset G$ be the associated parametrized elliptic maximal torus and let (T_1, B_1) be the associated fundamental pair. Let $\mathcal{D}_H$ be an arbitrary e.d. of $V^- \oplus V^+$ giving rise to the fixed orientations on $V^{\pm}$ and let $f_H : \mathrm{U}(1)^m \xrightarrow{\sim} T_H \subset H$ be the associated parametrized elliptic maximal torus. Let $j_1 := f_1 \circ f_H^{-1} : T_H \xrightarrow{\sim} T_1$. Then j_1 is admissible and we have*

$$\Delta_{j_1,B_1} \Delta_{\mathrm{Wh}}^{-1} = \begin{cases} (-1)^{\lfloor m^-/2 \rfloor}, & q = 0 \\ (-1)^{\lfloor m^-/2 \rfloor}, & q = 2. \end{cases}$$

Proof. This is a consequence of Proposition 4.7.9 and Remark 4.7.11. $\square$

4.7.13 Relation with cuspidal Levi subgroups

Consider an arbitrary $o_{\underline{V}}$-compatible e.d.

$$(V_1, \cdots, V_m)$$

of $\underline{V}$. Let $2 \leq n \leq m$ be an even integer such that among $V_1, \cdots, V_n$ exactly half of them are positive definite. Let

$$\underline{W} := V_{n+1} \oplus V_{n+2} \oplus \cdots \oplus V_m$$

equipped with the inherited quadratic form and orientation. Then $\mathrm{SO}(\underline{W})$ embeds into G^*, as a part of a Levi subgroup $M^* \subset G^*$ of the form $\mathrm{SO}(\underline{W})$ times a product of $\mathbb{G}_m$'s and GL_2's. Since for $\mathbb{G}_m$ or GL_2 there is a unique Whittaker datum (up to equivalence), $\mathrm{SO}(\underline{W})$ inherits a unique Whittaker datum from $\mathfrak{w}_I$ for G^*.

Proposition 4.7.14. *The Whittaker datum for* $\mathrm{SO}(\underline{W})$ *inherited from* $\mathfrak{w}_I$ *for* G^* *is the type I Whittaker datum for* $\mathrm{SO}(\underline{W})$.

Proof. We pick $\mathbb{R}$-pinnings $\mathbf{spl}_{\underline{W}}$ for $\mathrm{SO}(\underline{W})$ and $\mathbf{spl}$ for G^*, such that $\mathbf{spl}$ restricts to $\mathbf{spl}_{\underline{W}}$ in the sense that $\mathrm{SO}(\underline{W})$ multiplied with a product of $\mathbb{G}_m$'s and GL_2's is a Levi subgroup of G^*. In [81, §1.6], an invariant $\eta \in \mathbb{R}^\times/\mathbb{R}^{\times,2} \cong \{\pm 1\}$ is associated to $\mathbf{spl}$, which can be thought of as measuring the difference between $\mathbf{spl}$ and a standardly chosen pinning for the group $\mathrm{SO}(\underline{W})$. Similarly we get $\eta_{\underline{W}} \in \{\pm 1\}$ associated to $\mathbf{spl}_{\underline{W}}$. It is obvious from the definition that $\eta = \eta_{\underline{W}}$. Hence the proposition follows from the following theorem, applied to both $G^* = \mathrm{SO}(\underline{V})$ and $\mathrm{SO}(\underline{W})$. $\square$

The following result could be of independent interest.

Theorem 4.7.15. *The type I Whittaker datum* $\mathfrak{w}_I$ *for* $G^* = \mathrm{SO}(\underline{V})$ *is obtained from the one of the two* $G^*(\mathbb{R})$*-conjugacy class of* $\mathbb{R}$*-pinnings* $\mathbf{spl}_I$ *of* G^* *corresponding to* $\eta = -1 \in \{\pm 1\}$ *as in [81, §1.6].*

Proof. Consider the elliptic endoscopic datum $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (H, s, \eta)$ satisfying the cuspidality condition $\delta^\pm = (-1)^{d^\pm/2}$. Let $m^\pm := d^\pm/2$. Let Δ^*_{Wh} and $\tilde{\Delta}^*_{\mathrm{Wh}}$ be the transfer factors between H and G^* as in Definition 4.7.3. By the second statement in Lemma 4.7.4 we know that

$$\Delta^*_{\mathrm{Wh}} = (-1)^{m^-}\tilde{\Delta}^*_{\mathrm{Wh}}.$$

Hence it suffices to show that $\tilde{\Delta}^*_{\mathrm{Wh}}$ is equal to the Whittaker normalization $\Delta_{\mathfrak{w}_{\mathbf{spl}_I}}$ defined by the Whittaker datum $\mathfrak{w}_{\mathbf{spl}_I}$ defined by $\mathbf{spl}_I$, for one choice of (d^+, d^-) with $m^- = d^-/2$ odd. In the following we show that $\Delta^*_{\mathrm{Wh}} = \Delta_{\mathfrak{w}_{\mathbf{spl}_I}}$ without assuming m^- odd.

As in the setting of Lemma 4.7.4, we fix a type I Whittaker e.d. $\mathcal{D}^*$ of $\underline{V}$ and an arbitrary e.d. $\mathcal{D}_H = (\mathcal{D}_{H^+}, \mathcal{D}_{H^-})$ of V^- and V^+ induce the given orientations on V^- and V^+. Then we get an admissible isomorphism $j_{\mathcal{D}_H,\mathcal{D}^*} : T_{\mathcal{D}_H} \xrightarrow{\sim} T_{\mathcal{D}^*}$ as in the setting of that lemma. Let $B_{\mathcal{D}^*}$ be the Borel containing $T_{\mathcal{D}^*,\mathbb{C}}$ determined by $\mathcal{D}^*$ via (4.4.1). By Lemma 4.7.4, we have

$$\Delta^*_{\mathrm{Wh}} = \Delta_{j_{\mathcal{D}_H,\mathcal{D}^*}, B_{\mathcal{D}^*}}. \tag{4.7.1}$$

To simplify notation we denote $j := j_{\mathcal{D}_H,\mathcal{D}^*}$, $B := B_{\mathcal{D}^*}$, $T := T_{\mathcal{D}^*}$, $T_H := T_{\mathcal{D}_H}$. We also let $T^{\pm} := T_H \cap H^{\pm}$, and let B_H be the Borel of $H_{\mathbb{C}}$ containing $T_{H,\mathbb{C}}$ determined by (j, B). We now recall the explicit formula for $\Delta_{j,B}$ given in [40, §7] and recalled in [60, §3.2].[15] Let Λ be the set of B-positive roots for $(G^*_{\mathbb{C}}, T_{\mathbb{C}})$ not coming from H through j. Namely,

$$\Lambda = \left\{\epsilon_i + \epsilon_k, \epsilon_i - \epsilon_k | 1 \le i \le m^-,\ m^- + 1 \le k \le m\right\}.$$

Fix an element $\gamma \in T(\mathbb{R})$ and let $\gamma_H := j^{-1}(\gamma) \in T_H(\mathbb{R})$. For simplicity assume γ is strongly regular in G^*. Then we have

$$\Delta_{j,B}(\gamma_H, \gamma) = (-1)^{q(G^*)+q(H)}\chi_B(\gamma)\prod_{\alpha\in\Lambda}(1-\alpha(\gamma)) = \chi_B(\gamma)\prod_{\alpha\in\Lambda}(1-\alpha(\gamma)), \tag{4.7.2}$$

where χ_B is a quasi-character on $T(\mathbb{R})$ whose definition is recalled in [60, Définition 3.2.1].

[15] Note that there is a typo in [60, §3.2]. $(1-\alpha(\gamma^{-1}))$ there should be $1-\alpha(\gamma)$.

In [60, Exemple 3.2.4] Morel proves in a special case the following formula:

$$\chi_B = (\rho_{B_H} \circ j^{-1})\rho_B^{-1}, \tag{4.7.3}$$

where ρ_B and ρ_{B_H}, defined to be the half sum of the B-positive roots and the B_H-positive roots, are actual quasi-characters. In our case, ρ_B and ρ_{B_H} are also actual quasi-characters. We claim that (4.7.3) also holds in our case. In fact, when Morel proves (4.7.3), the only special property she uses is that the cocycle $a \in Z^1(W_{\mathbb{R}}, \hat{T})$ used to define χ_B could be arranged to satisfy the condition that it sends the element $\tau \in \mathbb{C}^\times \sqcup \mathbb{C}^\times\tau = W_{\mathbb{R}}$ to $1 \in \hat{T}$. This is not needed[16] in our case, because T is a product of $\mathrm{U}(1)$, and the image of a in $\mathbf{H}^1(W_{\mathbb{R}}, \hat{T})$, which determines χ_B via the local Langlands correspondence for T over $\mathbb{R}$, only depends on the homomorphism $a|_{W_{\mathbb{C}}} : W_{\mathbb{C}} \to \hat{T}$. The claim is proved.

By (4.7.3), we have

$$\chi_B = -m^+\epsilon_1 - m^+\epsilon_2 - \cdots - m^+\epsilon_{m^-}. \tag{4.7.4}$$

Having identified both T and T_H with $\mathrm{U}(1)^m$, we write

$$\gamma_H = \gamma = (y_1, y_2, \cdots, y_m)$$

with each $y_i \in \mathrm{U}(1)(\mathbb{R}) \subset \mathbb{C}^\times$. In conclusion, by (4.7.1) (4.7.2) (4.7.4), we have

$$\Delta^*_{\mathrm{Wh}}(\gamma_H, \gamma) = \prod_{\substack{1\le i\le m^- \\ m^-+1\le k\le m}} y_i^{-1}(1 - y_i y_k^{-1})(1 - y_i y_k) = \prod_{\substack{1\le i\le m^- \\ m^-+1\le k\le m}} 2(\Re y_i - \Re y_k). \tag{4.7.5}$$

We now compute $\Delta_{\mathfrak{w}_{\mathbf{spl}_I}}$. In [81] Waldspurger gives an explicit formula for the normalization Δ_0 (due to Langlands-Shelstad) without the factor Δ_{IV}. Let us denote Waldspurger's

[16]but could also be arranged.

formula by Δ_{Wal}. Hence we have (cf. [42, §5.3], [35, §5.5])

$$\Delta_{\mathfrak{w}_{\mathbf{spl}_I}} = \epsilon_L(V,\psi_{\mathbb{R}})\Delta_0 = \epsilon_L(V,\psi_{\mathbb{R}})\Delta_{\mathrm{Wal}}\Delta_{IV}, \tag{4.7.6}$$

where V is the virtual Γ_∞-representation $X^*(T_0)\otimes\mathbb{C} - X^*(T_{H,0})\otimes\mathbb{C}$, with T_0 inside $\mathbf{spl}_I$ and $T_{H,0}$ inside any $\mathbb{R}$-pinning of H. Since G^* is split, T_0 is necessarily split, so $X^*(T_0)$ is a direct sum of trivial representations of Γ_∞. When m^- is even, $X^*(T_{H,0})$ is a direct sum of trivial representations, and when m^- is odd, it is a direct sum of trivial representations and two copies of $X^*(\mathrm{U}(1))$. Therefore, by [74, (3.2.2), (3.6.1)] we have

$$\epsilon_L(V,\psi_{\mathbb{R}}) = \begin{cases} 1, & m^- \text{ even} \\ -1, & m^- \text{ odd} \end{cases} = (-1)^{m^-}. \tag{4.7.7}$$

By definition

$$\Delta_{IV}(\gamma_H,\gamma) = \prod_{\alpha\in\Lambda} |\alpha(\gamma)|^{-1/2}\,|1-\alpha(\gamma)| = \prod_{\substack{1\le i\le m^- \\ m^-+1\le k\le m}} 2\,|\Re y_i - \Re y_k|\,. \tag{4.7.8}$$

Waldspurger's explicit formula reads ([81, §1.10])

$$\Delta_{\mathrm{Wal}}(\gamma_H,\gamma) = \prod_{i=1}^{m^-} \mathrm{sgn}\left(\eta c_i(1+\Re y_i)\prod_{\substack{1\le k\le m \\ k\ne i}}(\Re y_i - \Re y_k)\right), \tag{4.7.9}$$

where $c_i \in \{\pm 1\}$ is the definiteness of V_i, namely $c_i = (-1)^{i+1}$. Note that $1+\Re y_i > 0$, and

$$\prod_{\substack{1\le i,k\le m^- \\ i\ne k}} \mathrm{sgn}(\Re y_i - \Re y_k) = (-1)^{m^-(m^--1)/2} = (-1)^{\lfloor m^-/2\rfloor},$$

$$\prod_{i=1}^{m^-} \operatorname{sgn} c_i = \prod_{i=1}^{m^-} (-1)^{i+1} = (-1)^{\lfloor m^-/2 \rfloor},$$

so from (4.7.9) we have

$$\Delta_{\mathrm{Wal}}(\gamma_H, \gamma) = \operatorname{sgn}(\eta)^{m^-} \prod_{\substack{1 \leq i \leq m^- \\ m^-+1 \leq k \leq m}} \operatorname{sgn}(\Re y_i - \Re y_k). \tag{4.7.10}$$

Combining (4.7.5) (4.7.6) (4.7.7) (4.7.8) (4.7.10) we have

$$\Delta_{\mathfrak{w}_{\mathbf{spl}_I}} = (-\operatorname{sgn}(\eta))^{m^-} \Delta^*_{\mathrm{Wh}}(\gamma_H, \gamma).$$

By assumption $\eta = -1$ for $\mathbf{spl}_I$, so the proof is completed. □

4.8 G-endoscopic data for Levi subgroups

In this subsection we specialize to the case where (V, q) is a quadratic space over $\mathbb{R}$ of signature $(d-2, 2) = (n, 2)$ as considered in §4.1. We will continue using the notations established there.

Denote $W_i := V_i^{\perp}/V_i, i = 1, 2$, considered as a quadratic space over $\mathbb{R}$. Recall that the signature of W_i is $(n-i, 2-i)$. We proceed similarly as before to determine the quasi-split inner form, L-group, and elliptic endoscopic data for $\mathrm{SO}(W_i)$. To fix notation we briefly go over the construction once more. Firstly define a quasi-split quadratic space $\underline{W_i}$ of the same dimension and discriminant as W_i. Then we fix an isomorphism

$$\phi_i : W_i \otimes \mathbb{C} \xrightarrow{\sim} \underline{W_i} \otimes \mathbb{C}$$

preserving the quadratic forms and satisfying the assumption in Definition 4.3.4, and use it to get an isomorphism

$$\psi_i : \mathrm{SO}(W_i)_{\mathbb{C}} \xrightarrow{\sim} \mathrm{SO}(\underline{W_i})_{\mathbb{C}}$$

$$g \mapsto \phi_i g \phi_i^{-1}.$$

Define the function

$$u_i : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{SO}(\underline{W_i})(\mathbb{C})$$

$$\rho \mapsto {}^{\rho}\phi_i \phi_i^{-1}.$$

We make two important compatibility assumptions. First note that it is always possible to embed $\underline{W}_i$ isometrically into $\underline{V}$, such that its orthogonal complement is a split quadratic space. We fix such an embedding. We also choose an embedding of M^{GL} into SO of the orthogonal complement of $\underline{W}_i$ in $\underline{V}$ as a Levi subgroup. Then we have an embedding $M^{\mathrm{GL}} \times \mathrm{SO}(\underline{W}_i) \hookrightarrow G^*$ as a Levi subgroup. We assume the following two conditions:

1. The following diagram commutes:

(4.8.1)
$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\psi_i"] & G \arrow[d, "\psi"] \\ M^{\mathrm{GL}} \times SO(\underline{W}_i) \arrow[r, hook] & \mathrm{SO}(\underline{V}) \end{tikzcd}$$

2. The function $u : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to G^*(\mathbb{C})$ is equal to the composition of u_i and the embedding $\mathrm{SO}(\underline{W}_i)(\mathbb{C}) \to \mathfrak{G}^*(\mathbb{C})$.

The fact that the two assumptions (together with the assumption in Definition 4.3.4 for both ϕ and ϕ_i) can be arranged can be easily seen by choosing suitable bases of $W_i, V, \underline{W}_i, \underline{V}$, and define the maps ϕ, ϕ_i on these bases.

Define the L-group ${}^L\mathrm{SO}(W_i)$ of $\mathrm{SO}(W_i)$ as in §4.3.7. If d is odd, we have

$${}^L\mathrm{SO}(W_i) = \mathrm{Sp}_{d-2i-1}(\mathbb{C}) \times W_{\mathbb{R}}.$$

If d is even, we have

$$^L\mathrm{SO}(W_i) = \mathrm{SO}_{d-2i}(\mathbb{C}) \rtimes W_{\mathbb{R}},$$

where the action of $W_{\mathbb{R}}$ is trivial or non-trivial according to whether the discriminant δ_i of W_i is 1 or -1. Note that for d even we have $\delta = \delta_1 = \delta_2$. Recall again that when d is even we need to fix an orientation on $\underline{W_i}$.

In general, we classify the elliptic endoscopic data for $\mathrm{SO}(W_i)$ as before, using pairs (d^+, d^-) or quadruples $(d^+, \delta^+, d^-, \delta^-)$, and for each such index we fix an explicit representative analogously as before. To specify that these data are for the group $\mathrm{SO}(W_i)$, we will use notations $e_{d^+,d^-}(W_i)$ or $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(W_i)$.

In the following we will work with the standard Levi subgroups M_1, M_2, M_{12}. We let S be a non-empty subset of $\{1, 2\}$, and use the notation M_S. We denote the decomposition (4.1.1) by

$$M_S = M_S^{\mathrm{GL}} \times M_S^{\mathrm{SO}},$$

where M_S^{SO} is $\mathrm{SO}(W_1)$ or $\mathrm{SO}(W_2)$. As we have already fixed the inner twisting to the quasi-split inner form of M_S^{SO}, we get an inner twisting to the quasi-split inner form of M_S by forming the direct product with M_S^{GL}. Moreover we form the L-group ^LM_S as

$$^LM_S = \widehat{M_S^{\mathrm{GL}}} \times (^LM_S^{\mathrm{SO}}),$$

where $\widehat{M_S^{\mathrm{GL}}}$ is given by

$$\widehat{\mathrm{GL}_k^{\times j}} = \mathrm{GL}_k(\mathbb{C})^{\times j}, \ \forall k, j \in \mathbb{N}.$$

Given an endoscopic datum for M_S^{SO}, we can extend it "trivially along M_S^{GL}" to get an endoscopic datum for M_S. Namely, we multiply the endoscopic group by M_S^{GL} and extend the other parts of the datum trivially. This induces a bijection from the isomorphism classes of elliptic endoscopic data for M_S^{SO} to those for M_S. Moreover the outer automorphism groups of the endoscopic data are preserved under this bijection. We will use notations $\mathfrak{e}_{d^+,d^-}(M_S)$ or $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M_S)$ to denote the elliptic endoscopic data for M_S obtained by extending the explicit representatives of the corresponding data for M_S^{SO} trivially along M_S^{GL}.

The concepts of endoscopic G-data for a Levi subgroup M of G, isomorphisms between them, and ellipticity, etc. are invented by Kottwitz in his unpublished notes and recalled in [59]. We use notations $\mathscr{E}(G), \mathscr{E}(M), \mathscr{E}_G(M)$ to denote the set of the elliptic endoscopic data for G up to isomorphism, the elliptic endoscopic data for M up to isomorphism, and the bi-elliptic G-endoscopic data for M up to isomorphism. Here bi-elliptic means that the underlying endoscopic datum for M and the induced endoscopic datum for G should both be elliptic.

In the following we construct representatives of the G-endoscopic data for M_S which exhaust all the isomorphism classes that are bi-elliptic. For $S = \{1\}$ or $\{1,2\}$ let $I_S := \{1,2\}$. For $S = \{2\}$ let $I_S := \{1\}$. Let $i_S = \max I_S \in \{1,2\}$. Then we have

$$M_S^{\mathrm{SO}} = \mathrm{SO}(W_{i_S}) = \mathrm{SO}(V_{i_S}^{\perp}/V_{i_S}).$$

We will consider a subset $\mathscr{P}_S$ of the power set $\mathscr{P}(I_S)$. For $S = \{1,2\}$ or $\{2\}$, let

$$\mathscr{P}_S := \mathscr{P}(I_S).$$

For $S = \{1\}$, let

$$\mathscr{P}_S := \{\{1,2\}, \emptyset\}.$$

We call the involution on $\mathscr{P}_S$ sending $A \in \mathscr{P}_S$ to its complement $A^c \in \mathscr{P}_S$ *swapping*. In the following we will fix S and omit it from all the subscripts.

The odd case.

Consider the set $\{(d^+, d^-) | d^+, d^- \text{ both odd}, d^+ + d^- = d + 1 - 2i\}$ parametrizing the elliptic endoscopic data for M (or, what amounts to the same, for M^{SO}). We will abbreviate this set simply as $\{(d^+, d^-)\}$. Form the direct product

$$\mathscr{P} \times \left\{(d^+, d^-)\right\}.$$

For each element (A, d^+, d^-) of this set, we construct a G-endoscopic datum for M as follows. From the pair (d^+, d^-) we build an (extended) endoscopic datum

$$((M')^{\mathrm{SO}}, s^{\mathrm{SO}}, \eta^{\mathrm{SO}} : {}^L(M')^{\mathrm{SO}} \to {}^L M^{\mathrm{SO}})$$

for $M^{\mathrm{SO}} = \mathrm{SO}(W_i)$ as before. We then define $(M', s_M, \eta_M : {}^L M' \to {}^L M)$, an endoscopic datum for M, as follows. Let $M' := M^{\mathrm{GL}} \times (M')^{\mathrm{SO}}$ as an algebraic group over $\mathbb{R}$. Let s_M be the element of ${}^L M = \widehat{M^{\mathrm{GL}}} \times {}^L M^{\mathrm{SO}}$ whose component in ${}^L M^{\mathrm{SO}}$ is s^{SO} and whose component in $\widehat{M^{\mathrm{GL}}}$ is

- the identity matrix of size $|A| = i$ if $A^c = \emptyset$,
- -1 times the identity matrix of size $|A^c| = i$ if $A = \emptyset$,
- $\mathrm{diag}(1, -1)$ if $A = \{1\}, A^c = \{2\}$,
- $\mathrm{diag}(-1, 1)$ if $A = \{2\}, A^c = \{1\}$.

Thus A (resp. A^c) is the subset of I recording the positions of 1's (resp. -1's) in the coordinates of the component of s_M in $\widehat{M^{\mathrm{GL}}}$.

We form the L-group ${}^L M'$ of M' as ${}^L M' = \widehat{M^{\mathrm{GL}}} \times {}^L(M')^{\mathrm{SO}}$. The map

$$\eta_M : {}^L M' = \widehat{M^{\mathrm{GL}}} \times {}^L(M')^{\mathrm{SO}} \to {}^L M = \widehat{M^{\mathrm{GL}}} \times {}^L M^{\mathrm{SO}}$$

is given by trivially extending η^{SO}. Thus we have defined an endoscopic datum (M', s_M, η_M) for M_S. It turns out to be a G-endoscopic datum for M and is well defined up to isomorphisms between G-endoscopic data for M. It also turns out to be bi-elliptic. In the future, we will use the notation $\mathfrak{e}_{A,A^c,d^+,d^-}(M,G) = \mathfrak{e}_{A,d^+,d^-}$ to denote the particular representative of the G-endoscopic datum for M as constructed above, rather than just its isomorphism class.

In this way the set $\mathscr{P} \times \{(d^+,d^-)\}$ parametrizes all the bi-elliptic G-endoscopic data for M up to isomorphism. Two unequal elements (A,d^+,d^-) and (A_1,d_1^+,d_1^-) parametrize isomorphic G-endoscopic data for M if and only if $A_1 = A^c$ and $d^+ = d_1^-$. In other words, we have a bijection

$$\mathscr{P} \times \left\{(d^+,d^-)\right\} /\text{simultaneous swapping} \xrightarrow{\sim} \mathscr{E}_G(M). \tag{4.8.2}$$

The maps $\mathscr{E}_G(M) \to \mathscr{E}(G)$ and $\mathscr{E}_G(M) \to \mathscr{E}(M)$ now can be described in terms of the relevant parameters. The first map is

$$\mathfrak{e}_{A,d^+,d^-}(M,G) \mapsto \mathfrak{e}_{d^++2|A|,d^-+2|A^c|}$$

and the second map is

$$\mathfrak{e}_{A,d^+,d^-}(M,G) \mapsto \mathfrak{e}_{d^+,d^-}(M).$$

We provide one more piece of information about the map $\mathscr{E}_G(M) \to \mathscr{E}(G)$. Consider the G-endoscopic datum

$$\mathfrak{e}_{A,d^+,d^-}(M,G) = (M', s_M, \eta_M : {}^LM' \to {}^LM)$$

for M. We have seen that up to isomorphism, it determines the endoscopic datum

$$\mathfrak{e}_{d^++2|A|,d^-+2|A^c|} = (H, s, \eta : {}^LH \to {}^LG)$$

for G. Recall that $H = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$, where V^+ (resp. V^-) is a split quadratic space of dimension $d^+ + 2|A|$ (resp. $d^- + 2|A^c|$). Also

$$M' = M^{\mathrm{GL}} \times \mathrm{SO}(W^+) \times \mathrm{SO}(W^-),$$

where $W^\pm$ is a split quadratic space of dimension $d^\pm$. Now there is a canonical way of viewing M' as a Levi subgroup of H, up to $H(\mathbb{R})$-conjugacy. We describe it now. Firstly we choose isometric embeddings of $W^\pm$ into $V^\pm$. For simplicity we now assume $|A| \geq |A^c|$. The other cases are symmetric to these cases.

Suppose $A = \{1, 2\}$ and $A^c = \emptyset$. We have $M^{\mathrm{GL}} = \mathrm{GL}_2$ or $\mathbb{G}_m^2$. First suppose $M^{\mathrm{GL}} = \mathrm{GL}_2$. We choose a decomposition

$$V^+ = W^+ \oplus p \oplus p^\vee,$$

where p is an isotropic subspace of dimension i and W^+ is orthogonal to $p \oplus p^\vee$. We embed $M^{\mathrm{GL}} \times \mathrm{SO}(W^+)$ into $\mathrm{SO}(V^+)$ as the stabilizer of this decomposition of V^+. The chosen embedding $W^- \hookrightarrow V^-$ is an isomorphism, so we have an isomorphism $\mathrm{SO}(W^-) \xrightarrow{\sim} \mathrm{SO}(V^-)$. Together we get an embedding $M' \to H$. Now we treat the case $M^{\mathrm{GL}} = \mathbb{G}_m^2$. This time we choose a decomposition

$$V^+ = W^+ \oplus l_1 \oplus l_1^\vee \oplus l_2 \oplus l_2^\vee,$$

where l_1 and l_2 are isotropic lines and all three of $W^+, (l_1 \oplus l_1^\vee), (l_2 \oplus l_2^\vee)$ are mutually orthogonal. We embed $M^{\mathrm{GL}} \times \mathrm{SO}(W^+)$ into $\mathrm{SO}(V^+)$ as the stabilizer of this decomposition of V^+. Again the chosen embedding $W^- \hookrightarrow V^-$ is an isomorphism, so we have an isomorphism $\mathrm{SO}(W^-) \xrightarrow{\sim} \mathrm{SO}(V^-)$. Together we get an embedding $M' \to H$

Suppose $A = \{1\}$ and $A^c = \emptyset$. In this case $M^{\mathrm{GL}} = \mathbb{G}_m$. We choose a decomposition

$$V^+ = W^+ \oplus l \oplus l^\vee$$

where l is an isotropic line and W^+ is orthogonal to $l \oplus l^\vee$. We embed $M^{\mathrm{GL}} \times \mathrm{SO}(W^+)$ into

$\mathrm{SO}(V^+)$ as the stabilizer of this decomposition of V^+. The chosen embedding $W^- \hookrightarrow V^-$ is an isomorphism, so we have an isomorphism $\mathrm{SO}(W^-) \xrightarrow{\sim} \mathrm{SO}(V^-)$. Together we get an embedding $M' \to H$.

Suppose $A = \{k^+\}$ and $A^c = \{k^-\}$, with $\{k^+, k^-\} = \{1, 2\}$. For V^+ and V^-, we choose a decomposition

$$V^\pm = W^\pm \oplus l^\pm \oplus l^{\pm,\vee}$$

where $l^\pm$ is an isotropic line and $W^\pm$ is orthogonal to $l^\pm \oplus l^{\pm,\vee}$. The stabilizers of these decompositions define Levi subgroups of $\mathrm{SO}(V^\pm)$ of the form

$$M^\pm = \mathbb{G}_m \times \mathrm{SO}(W^\pm).$$

The desired map

$$M^{\mathrm{GL}} \times \mathrm{SO}(W^+) \times \mathrm{SO}(W^-) = \mathbb{G}_m^2 \times \mathrm{SO}(W^+) \times \mathrm{SO}(W^-) \to \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$$

is given by grouping the $k^\pm$-th factor of $\mathbb{G}_m^2$ with $\mathrm{SO}(W^\pm)$ and then mapping it into $\mathrm{SO}(V^\pm)$ as the above-mentioned Levi subgroup.

Finally we determine the G-outer automorphism groups of elements in $\mathscr{E}_G(M)$. We use the notation $\mathrm{Out}_G(\cdot)$. Let $\mathfrak{e} = \mathfrak{e}_{A,d^+,d^-}(M, G)$. Note that by definition, $\mathrm{Out}_G(\mathfrak{e})$ is a subgroup of $\mathrm{Out}(\mathfrak{e}_{d^+,d^-}(M))$. The latter group, equal to $\mathrm{Out}(\mathfrak{e}_{d^+,d^-}(W_i))$ (where we recall that $\mathfrak{e}_{d^+,d^-}(W_i)$ is is an endoscopic datum for $\mathrm{SO}(W_i)$), is nontrivial only for $d^+ = d^-$, in which case it contains one non-trivial element, which switches the two factors of the endoscopic group. This extra outer automorphism sends the element $s_M = (s^{\mathrm{GL}} s^{\mathrm{SO}})$ in $\mathfrak{e}$ to $(s^{\mathrm{GL}}, -s^{\mathrm{SO}})$. The two elements $(s^{\mathrm{GL}}, \pm s^{\mathrm{SO}})$ are equal modulo $Z(\widehat{M}) = Z(\widehat{M^{\mathrm{GL}}}) \times Z(\widehat{M^{\mathrm{SO}}})$, but they are not equal modulo $Z(\hat{G})$ (after being mapped into $\hat{G}$ under an admissible embedding ${}^L M \to {}^L G$.) Therefore this outer automorphism is not a G-outer automorphism. We conclude that $\mathrm{Out}_G(\mathfrak{e}) = 1$ for any $\mathfrak{e} \in \mathscr{E}_G(M)$.

The even case. In the even case, we first need to construct

Consider the set

$$\left\{(d^+, \delta^+, d^-, \delta^-) | d^+, d^- \text{ both even}, \ d^+ + d^- = d - 2i, \ \delta^+\delta^- = \delta\right\} \text{ minus}$$

$$\{(0, -1), (2, 1)\} \times \{(0, -1), (2, 1)\}$$

parametrizing the elliptic endoscopic data for M_S. (Recall that in the even case both W_1 and W_2 have discriminant δ.) We will abbreviate this set simply as $\{(d^+, \delta^+, d^-, \delta^-)\}$. Similarly as before we get a parametrization of all the bi-elliptic G-endoscopic data for M up to isomorphism, by the set

$$\mathscr{P} \times' \left\{(d^+, \delta^+, d^-, \delta^-)\right\},$$

defined to be the subset of the Cartesian product consisting of tuples $(A, d^+, \delta^+, d^-, \delta^-)$ such that none of the following pairs is equal to $(0, -1)$ or $(2, 1)$:

$$(d^+, \delta^+), \quad (d^-, \delta^-), \quad (d^+ + 2|A|, \delta^+), \quad (d^- + 2|A^c|, \delta^-).$$

We will use the notation $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M, G)$ to to denote the corresponding G-endoscopic datum for M_S. As in the odd case, we understand that this notation means the explicit representative constructed analogously as in the odd case, rather than just its isomorphisms class. We have an induced bijection

$$\mathscr{P} \times' \left\{(d^+, \delta^+, d^-, \delta^-)\right\} / \text{simultaneous swapping} \xrightarrow{\sim} \mathscr{E}_G(M). \tag{4.8.3}$$

The map $\mathscr{E}_G(M) \to \mathscr{E}(M)$ is given by

$$\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M,G) \mapsto \mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M).$$

The map $\mathscr{E}_G(M) \to \mathscr{E}(G)$ is given by

$$\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M,G) \mapsto \mathfrak{e}_{d^++2|A|,\delta^+,d^-+2|A^c|,\delta^-}.$$

As before, the endoscopic group M' in the datum $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M,G)$ can be viewed as a Levi subgroup of the endoscopic group H in the datum $\mathfrak{e}_{d^++2|A|,\delta^+,d^-+2|A^c|,\delta^-}$, in a way that is canonical up to conjugation by $H(\mathbb{R})$. The description of this is the same as in the odd case. The only new feature is that, with the same notation as in the corresponding discussion in the odd case, because $V^\pm$ has the same discriminant as $W^\pm$, it is possible to isometrically embed $W^\pm$ into $V^\pm$ in such a way that its orthogonal complement is a split quadratic space.

We now determine the G-outer automorphism groups of elements in $\mathscr{E}_G(M)$. Recall from §4.3.11 that there are two kinds of possible nontrivial outer automorphisms of the endoscopic datum for M underlying $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M,G)$. One is swapping the two factors and the other is simultaneous nontrivial outer automorphisms on the two factors of the endoscopic group. The former is never a G-outer automorphism, by an argument similar to the odd case. The latter, when present, is a G-outer automorphism. We conclude that

$$\mathrm{Out}_G(\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}) = \begin{cases} 1, & d^+d^- = 0 \\ \mathbb{Z}/2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

4.9 Kostant-Weyl traces and discrete series characters, case M_1

We keep the setting in §4.1. Fix an irreducible algebraic representation $\mathbb{V}$ of $G_{\mathbb{C}}$. This gives rise to an L-packet $\Pi(\mathbb{V})$ of discrete series representations of $G(\mathbb{R})$. Let Θ be the stable character associated to $\Pi(\mathbb{V})$. i.e. Θ is the sum of the characters of members of $\Pi(\mathbb{V})$. Note that we follow [60] in this convention, whereas in [18] a factor $(-1)^{q(G)}$ is included in the definition of Θ. Consider $M := M_1$, isomorphic to

$$\mathrm{GL}_2 \times \mathrm{SO}(V_2^{\perp}/V_2) \cong \mathrm{GL}_2 \times \mathrm{SO}(d-4,0).$$

The normalized stable discrete series character $\Phi_M^G(\gamma, \Theta)$ for $\gamma \in M(\mathbb{R})$ is studied in [1] and [18]. See also [59, §3.2].

For our purpose it suffices to only consider regular elements $\gamma \in T_1(\mathbb{R})$, where T_1 is the elliptic maximal torus of M fixed in §4.1. Here regularity is understood with respect to the root system of $(G_{\mathbb{C}}, T_1)$ (not $(M_{\mathbb{C}}, T_1)$). Recall that T_1 is of the form

$$T_1 = T_{\mathrm{GL}_2}^{\mathrm{std}} \times T_{V_2^{\perp}/V_2}.$$

We have also fixed in §4.1 a parametrized maximal split torus $T' \cong \mathbb{G}_m^m$ of $M_{\mathbb{C}}$ (where $m = \lfloor d/2 \rfloor$), and an inner automorphism ι_1 of $M_{\mathbb{C}}$ that maps T_1 to T'. We have denoted the standard characters of T' by $\epsilon_1, \cdots, \epsilon_m$ and used the same symbols to denote the corresponding characters of $T_{1,\mathbb{C}}$, identifying $X^*(T_1)$ with $X^*(T')$ using ι_1. Recall from §4.1 that the real roots are $R = \{\pm(\epsilon_1 + \epsilon_2)\}$ in both the odd and the even case.

We write

$$\gamma = (\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \gamma_{V_2^{\perp}/V_2}) \in T_1(\mathbb{R}) = T_{\mathrm{GL}_2}^{\mathrm{std}}(\mathbb{R}) \times T_{V_2^{\perp}/V_2}(\mathbb{R}),$$

with $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Note that the value $(\epsilon_1 + \epsilon_2)(\gamma)$ is just the determinant on the GL_2 factor, namely $a^2 + b^2$.

We fix the positive system in the (complex) root datum $\Phi(G_{\mathbb{C}}, T') \cong \Phi(G_{\mathbb{C}}, T_1)$ corresponding to the natural order. Denote by B the corresponding Borel subgroup of $G_{\mathbb{C}}$ containing T_1. This choice also endows the root datum $\Phi(M_{1,\mathbb{C}}, T_1)$ with a positive system.

Denote:

- $\Phi^+ :=$ positive roots in $\Phi(G_{\mathbb{C}}, T_1)$.
- $\Phi^+_M :=$ positive roots in $\Phi(M_{\mathbb{C}}, T_1)$.
- $\rho \in X^*(T_1) \otimes \mathbb{Q}$ the half sum of the positive roots in $\Phi(G_{\mathbb{C}}, T_1)$.
- $\lambda \in X^*(T_1)$ the highest weight of $\mathbb{V}$.
- $N_1 :=$ the unipotent radical of P_1.
- $\varpi_1 := \epsilon_1^\vee + \epsilon_2^\vee \in X_*(T_1)$, and $t_1 = 1 - (d-2) = 3 - d$, as in Definition 2.2.1.
- $\Omega :=$ the complex Weyl group for $(G_{\mathbb{C}}, T_1)$.

Recall that L_M is defined in Definition 2.4.1.

Proposition 4.9.1. *Suppose $a^2 + b^2 < 1$. Then we have*

$$\Phi^G_M(\gamma, \Theta) = 2(-1)^{q(G)+1} L_M(\gamma). \tag{4.9.1}$$

Proof. We first compute the LHS. We follow the notations of [60]. By [60, Fait 3.1.6], which follows [18, §4], it is equal to

$$(-1)^{q(G)} \epsilon_R(\gamma) \delta_{P_1(\mathbb{R})}(\gamma)^{1/2} \Delta_M(\gamma)^{-1} \sum_{\omega \in \Omega} \epsilon(\omega) n(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma). \tag{4.9.2}$$

In our case, T_1 is isomorphic to $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times \mathrm{U}(1)^{m-2}$, so $T_1(\mathbb{R})$ is connected. Let A_1 be the maximal split sub-torus of T_1. Then $A_1 = \mathbb{G}_m$, diagonally embedded inside $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. The maximal compact subgroup $T_1(\mathbb{R})_1$ of $T_1(\mathbb{R})$ is $U(1)(\mathbb{R}) \times U(1)(\mathbb{R})^{m-2}$, where the first $U(1)(\mathbb{R})$ lies inside $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)(\mathbb{R}) = \mathbb{C}^\times$. We have the decomposition

$$T_1(\mathbb{R}) = A_1(\mathbb{R})^0 \times T_1(\mathbb{R})_1,$$

under which the projection of γ to $A_1(\mathbb{R})^0$ equals to $\exp(x) = \sqrt{a^2+b^2}$, where

$$x = \log \sqrt{a^2+b^2} \in \mathbb{R} \cong \mathrm{Lie}(A_1) = X_*(A_1)_{\mathbb{R}}.$$

Let's explain some notations.

$$\delta_{P_1(\mathbb{R})}(\gamma) := |\det(\mathrm{ad}(\gamma), \mathrm{Lie}(N_1))|_{\mathbb{R}}$$

$$\Delta_M(\gamma) := \prod_{\alpha \in \Phi_M^+} (1 - \alpha^{-1}(\gamma)).$$

Also we have

$$R_\gamma := \{\alpha \in R | \alpha(\gamma) > 0\} = R.$$

Let $R_\gamma^+ = \{\alpha \in R | \alpha(\gamma) > 1\}$. Namely, we have $R_\gamma^+ = \{-\epsilon_1 - \epsilon_2\}$. Define

$$\epsilon_R(\gamma) := (-1)^{\left|\Phi^+ \cap (-R_\gamma^+)\right|}.$$

In other words, $\epsilon_R(\gamma)$ is -1 to the number of positive roots in R that send γ into $]0,1[$. We have

$$\epsilon_R(\gamma) = (-1)^1 = -1.$$

For $\omega \in \Omega$, define

- $\Phi(\omega) = \Phi^+ \cap (-\omega\Phi^+)$,
- $l(\omega) = |\Phi(\omega)|$,
- $\epsilon(\omega) = (-1)^{l(\omega)}$.

Now we recall the definition of $n(\gamma, \omega B)$. Since $T_1(\mathbb{R})$ is connected, we have $\gamma \in G(\mathbb{R})^0$, and $n(\gamma, \omega B)$ can be computed using the *discrete series constants*. (If γ were not in

$Z_G(\mathbb{R})G(\mathbb{R})^0 = Z_G(\mathbb{R})\operatorname{im}(G^{\mathrm{SC}}(\mathbb{R}))$, we would have $n(\gamma, \omega B) = 0$.) As the root system $R = R_\gamma$ in $X^*(A_1)_{\mathbb{R}}$ has its Weyl group $\{\pm 1\}$ containing -1, we have the "discrete series constant" function:

$$\bar{c} = \bar{c}_R : X_*(A_1)_{\mathbb{R}} \times X^*(A_1)_{\mathbb{R}} \to \mathbb{Z}.$$

For the general definition of the discrete series constants cf. [18, §3]. We have

$$n(\gamma, \omega B) = \bar{c}(x, \wp(\omega\lambda + \omega\rho)).$$

In our case, we have assumed $a^2 + b^2 < 1$, so

$$x = \log\sqrt{a^2+b^2} \in \mathbb{R}_{<0} \cong \mathbb{R}_{>0}(-\epsilon_1^\vee - \epsilon_2^\vee).$$

Hence

$$n(\gamma, \omega B) = \begin{cases} 2, & \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1 + \epsilon_2) \\ 0, & \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(-\epsilon_1 - \epsilon_2). \end{cases}$$

We conclude that

(4.9.3)

$$\Phi_M^G(\gamma, \Theta) = 2(-1)^{q(G)+1}\delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\substack{\omega\in\Omega \\ \wp(\omega(\lambda+\rho))\in\mathbb{R}_{>0}(\epsilon_1+\epsilon_2)}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).$$

Now we compute the RHS of (4.9.1). Let Ω_1 be the Weyl group of $(M_{\mathbb{C}}, T_1)$. It is naturally a subgroup of Ω. Consider

$$\Omega_1' := \{\omega \in \Omega | \Phi(\omega) \subset \Phi(T_1, N_1)\} \subset \Omega.$$

It is a set of representatives of $\Omega_1 \backslash \Omega$. We now apply Kostant's Theorem ([33]), in the form as on p. 1700 of [60]. This states that as an algebraic representation of $M_{\mathbb{C}}$, we have

$$\mathbf{H}^k(\mathrm{Lie}(N_1), \mathbb{V}) \cong \bigoplus_{\substack{\omega' \in \Omega'_1 \\ l(\omega')=k}} \mathbb{V}_{\omega'(\lambda+\rho)-\rho}.$$

Here $\mathbb{V}_{\omega'(\lambda+\rho)-\rho}$ denotes the irreducible algebraic representation of $M_{\mathbb{C}}$ whose highest weight with respect to the fixed positive system in $\Phi(M_{1,\mathbb{C}}, T_1)$ is $\omega'(\lambda+\rho)-\rho$. Consequently,

$$\mathbf{H}^k(\mathrm{Lie}(N_1), \mathbb{V})_{>t_1} \cong \sideset{}{'}\bigoplus_{\substack{\omega' \in \Omega'_1 \\ l(\omega')=k \\ \langle \omega'(\lambda+\rho)-\rho, \varpi_1 \rangle > t_1}} \mathbb{V}_{\omega'(\lambda+\rho)-\rho}.$$

As we have

$$t_1 = \langle -\rho, \varpi_1 \rangle \tag{4.9.4}$$

by a simple computation, the condition that $\langle \omega'(\lambda+\rho)-\rho, \varpi_1 \rangle > t_1$ is equivalent to

$$\langle \omega'(\lambda+\rho), \varpi_1 \rangle > 0.$$

According to Weyl's character formula (cf. [60, Fait 3.1.6]), we have

$$\mathrm{Tr}(\gamma, \mathbb{V}_{\lambda'}) = \Delta_M(\gamma)^{-1} \sum_{\omega_1 \in \Omega_1} \epsilon(\omega_1)(\omega_1 \lambda')(\gamma) \prod_{\alpha \in \Phi_M(\omega_1)} \alpha^{-1}(\gamma),$$

where $\Phi_M(\omega_1) := \Phi_M^+ \cap (-\omega_1 \Phi_M^+)$ is defined analogously as $\Phi(\omega)$. Therefore we have

$$\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}(N_1), \mathbb{V})_{>t_1}) = \sum_{\substack{\omega' \in \Omega_1' \\ \langle \omega'(\lambda+\rho), \varpi_1 \rangle > 0}} (-1)^{l(\omega')} \mathrm{Tr}(\gamma, \mathbb{V}_{\omega'(\lambda+\rho)-\rho})$$

$$= \sum_{\substack{\omega' \in \Omega_1' \\ \langle \omega'(\lambda+\rho), \varpi_1 \rangle > 0}} \epsilon(\omega') \Delta_M(\gamma)^{-1} \sum_{\omega_1 \in \Omega_1} \epsilon(\omega_1)(\omega_1(\omega'(\lambda+\rho)-\rho))(\gamma) \prod_{\alpha \in \Phi_M(\omega_1)} \alpha^{-1}(\gamma).$$

Since ϖ_1 are invariant under Ω_1, and multiplication induces a bijection $\Omega_1 \times \Omega_1' \to \Omega$, the last formula is equal to

$$\sum_{\substack{\omega \in \Omega \\ \langle \omega(\lambda+\rho), \varpi_1 \rangle > 0}} \epsilon(\omega) \Delta_M(\gamma)^{-1} (\omega\lambda)(\gamma)(\omega\rho - p_1(\omega)\rho) \prod_{\alpha \in \Phi_M(p_1(\omega))} \alpha^{-1}(\gamma),$$

where for $\omega \in \Omega$ we have denoted by $p_1(\omega)$ the unique element in Ω_1 such that $\omega = p_1(\omega)\omega'$ for some $\omega' \in \Omega_1'$. Now for any three elements $\omega_1 \in \Omega_1, \omega' \in \Omega_1', \omega \in \Omega$ satisfying $\omega = \omega_1\omega'$, we have the relation

$$\omega(\rho) - \omega_1(\rho) - \sum_{\alpha \in \Phi_M(\omega_1)} \alpha = - \sum_{\alpha \in \Phi(\omega)} \alpha.$$

We conclude that

$$\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}(N_1), \mathbb{V})_{>t_1}) = \sum_{\omega \in \Omega, \langle \omega(\lambda+\rho), \varpi_1 \rangle > 0} \epsilon(\omega) \Delta_M(\gamma)^{-1} (\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma),$$

so by Definition 2.4.1,

$$L_M(\gamma) = \delta_{P_1(\mathbb{R})}^{1/2} \sum_{\omega \in \Omega, \langle \omega(\lambda+\rho), \varpi_1 \rangle > 0} \epsilon(\omega) \Delta_M(\gamma)^{-1} (\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma). \tag{4.9.5}$$

Comparing (4.9.3) and (4.9.5), we see that the proof is complete once we check that for

$\omega \in \Omega$,

$$\langle \omega(\lambda+\rho), \varpi_1 \rangle > 0 \Longleftrightarrow \wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1+\epsilon_2).$$

Since either $\wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(\epsilon_1+\epsilon_2)$ or $\wp(\omega(\lambda+\rho)) \in \mathbb{R}_{>0}(-\epsilon_1-\epsilon_2)$, the above equivalence is true. $\square$

4.10 Kostant-Weyl traces and discrete series characters, odd case M_2

In the even case the following discussion on M_2 is not needed since $\mathrm{Tr}_{M_2} = 0$ (cf. Remark 2.6.2). We thus assume that d is odd.

Let $M := M_2$, isomorphic to

$$\mathbb{G}_m \times \mathrm{SO}(V_1^{\perp}/V_1).$$

Let T_2 be as in §4.1. Let Θ be as in §4.9. We will consider the values $\Phi_M^G(\gamma, \Theta)$, for regular $\gamma \in T_2(\mathbb{R})$. Recall from §4.1 that

$$T_2 = \mathbb{G}_m \times T_{V_1^{\perp}/V_1}.$$

Accordingly we write

$$\gamma = (a, \gamma_{V_1^{\perp}/V_1}), \quad a \in \mathbb{R}^{\times}$$

Recall from §4.1 that the real roots of $(G_{\mathbb{C}}, T_2)$ are $R = \{\pm\epsilon_1\}$.

Denote

$$\varpi_2 := 2\epsilon_1^{\vee} \in X_*(T_{12})$$

$$t_2 := 0 - (d-2) = 2 - d$$

as in Definition 2.2.1.

Recall that $L_M(\gamma)$ is defined in Definition 2.4.2.

Proposition 4.10.1. *When $a < 0$, we have $\Phi_M^G(\gamma, \Theta) = 0$. When $0 < a < 1$, we have*

$$\Phi_M^G(\gamma, \Theta) = (-1)^{q(G)+1} L_M(\gamma).$$

Proof. As in the proof of Proposition 4.9.1, we have the following formula for $\Phi_M^G(\gamma, \Theta)$:

$$\Phi_M^G(\gamma, \Theta) = (-1)^{q(G)} \epsilon_R(\gamma) \delta_{P_1(\mathbb{R})}(\gamma)^{1/2} \Delta_M(\gamma)^{-1} \sum_{\omega \in \Omega} \epsilon(\omega) n(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma).$$

Now T_2 is isomorphic to $\mathbb{G}_m \times \mathrm{U}(1)^{m-1}$. Let A_2 be the maximal split sub-torus of T_2. Then $A_2 = \mathbb{G}_m$. The maximal compact subgroup $T_2(\mathbb{R})_1$ of $T_2(\mathbb{R})$ is $\{\pm 1\} \times U(1)(\mathbb{R})^{m-1}$. We have the decomposition

$$T_2(\mathbb{R}) = A_2(\mathbb{R})^0 \times T_2(\mathbb{R})_1,$$

under which the projection of γ to $A_2(\mathbb{R})^0$ equals to $\exp(x) = |a|$, where

$$x = \log|a| \in \mathbb{R} \cong \mathrm{Lie}(A_2) = X_*(A_2)_{\mathbb{R}}.$$

When $a < 0$, we have $R_\gamma = \emptyset$, so its Weyl group in $X_*(A_2)$ does not contain -1. Then as argued in [18, §4], γ cannot lie inside $Z_G(\mathbb{R}) \operatorname{im}(G^{\mathrm{SC}}(\mathbb{R})) = G(\mathbb{R})^0$ (which can also be seen directly), and hence all $n(\gamma, \omega B) = 0$.

Assume $0 < a < 1$. Then $\gamma \in G(\mathbb{R})^0$. Thus the numbers $n(\gamma, \omega B)$ are computed using the root system $R_\gamma = R = \{\pm\epsilon_1\}$ in $X^*(A_2)_{\mathbb{R}} = \mathbb{R}$.

We have

$$\epsilon_R(\gamma) = -1.$$

$$n(\gamma,\omega B)=\bar{c}(x,\wp(\omega\lambda+\omega\rho))=\begin{cases}2, & \wp(\omega(\lambda+\rho))\in\mathbb{R}_{>0}\epsilon_1\\ 0, & \wp(\omega(\lambda+\rho))\in\mathbb{R}_{>0}(-\epsilon_1).\end{cases}$$

Note that analogously to (4.9.4), we have

$$t_2=\langle-\rho,\varpi_2\rangle. \tag{4.10.1}$$

Using this, a similar computation as (4.9.5) yields

$$\operatorname{Tr}(\gamma,R\Gamma(\operatorname{Lie}(N_2),\mathbb{V})_{>t_2})=\sum_{\substack{\omega\in\Omega\\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}}\epsilon(\omega)\Delta_M(\gamma)^{-1}(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma),$$

where Ω is the complex Weyl group for $(G_{\mathbb{C}},T_2)$. Regarding Definition 2.4.2, we are reduced to checking that for $\omega\in\Omega$,

$$\langle\omega(\lambda+\rho),\varpi_2\rangle>0\Longleftrightarrow\wp(\omega(\lambda+\rho))\in\mathbb{R}_{>0}\epsilon_1,$$

which is obvious. $\square$

4.11 Kostant-Weyl traces and discrete series characters, case M_{12}

In this subsection we drop the assumption that d is odd made in §4.10. Let $M:=M_{12}$, isomorphic to

$$\mathbb{G}_m\times\mathbb{G}_m\times\operatorname{SO}(V_2^{\perp}/V_2).$$

Let T_{12} be as in §4.1. Let Θ be the same as in §4.9 and §4.10. We will consider the values $\Phi_M^G(\gamma,\Theta)$, for regular $\gamma\in T_{12}(\mathbb{R})$. In fact, we will also consider a variant of it, denoted by

$$\Phi_M^G(\gamma,\Theta)_{\mathrm{endosc}}.$$

Recall from §4.1 that

$$T_2 = \mathbb{G}_m \times \mathbb{G}_m \times T_{V_2^{\perp}/V_2}.$$

Accordingly we write

$$\gamma = (a, b, \gamma_{V_1^{\perp}/V_1}), \quad a, b \in \mathbb{R}^{\times}.$$

Recall from §4.1 that the real roots of $(G_{\mathbb{C}}, T_{12})$ are

$$\begin{cases} R = \{\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}, & \text{odd case} \\ R = \{\pm\epsilon_1 \pm \epsilon_2\}, & \text{even case.} \end{cases}$$

As in §4.9, we fix the positive system in $\Phi(G_{\mathbb{C}}, T') \cong \Phi(G_{\mathbb{C}}, T_{12})$ corresponding to the natural order. Denote by B the corresponding Borel subgroup of $G_{\mathbb{C}}$ containing T_{12}. This choice also endows the root datum $\Phi(M_{12,\mathbb{C}}, T_{12})$ with a positive system.

Denote:

- $\Phi^+ :=$ positive roots in $\Phi(G_{\mathbb{C}}, T_{12})$.
- $\Phi^+_M :=$ positive roots in $\Phi(M_{\mathbb{C}}, T_{12})$.
- $\Phi^+_{M_2} :=$ positive roots in $\Phi(M_{2,\mathbb{C}}, T_2)$. (This can be viewed also as a subset of $X^*(T_{12})$, as we have fixed isomorphisms $T_{12} \xrightarrow{\sim} T' \xrightarrow{\sim} T_2$ over $\mathbb{C}$.)
- $\rho \in X^*(T_{12}) \otimes \mathbb{Q}$ the half sum of the positive roots in $\Phi(G_{\mathbb{C}}, T_{12})$.
- $\lambda \in X^*(T_{12})$ the highest weight of $\mathbb{V}$.
- $\varpi_1, \varpi_2, t_1, t_2$ as in §4.9 and §4.10, or Definition 2.2.1.
- $\Omega :=$ the complex Weyl group for $(G_{\mathbb{C}}, T_{12})$.

Recall that $L_M(\gamma)$ is defined in Definition 2.3.5.

Lemma 4.11.1. *We have*

$$L_M(\gamma) = \delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma)\,\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}\,N_{P_{12}}, \mathbb{V})_{>t_{12}}) + \delta^{1/2}_{P_{12}(\mathbb{R})}(n_{12}\gamma n_{12}^{-1})\,\mathrm{Tr}(n_{12}\gamma n_{12}^{-1}, R\Gamma(\mathrm{Lie}\,N_{P_{12}}, \mathbb{V})_{>t_{12}}) - \left|D^{M_2}_M(\gamma)\right|^{1/2}\delta^{1/2}_{P_2(\mathbb{R})}(\gamma)\,\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}\,N_{P_2}, \mathbb{V})_{>t_2}). \tag{4.11.1}$$

Proof. From Definition 2.3.5 and Lemma 2.3.3, it suffices to compute the two quantities

$$(-1)^{\dim A_{gMg^{-1}}/A_{M_P}}$$

$$(n^{M_P}_{gMg^{-1}})^{-1} := \left|\mathrm{Nor}_{M_P}(gMg^{-1})(\mathbb{Q})/(gMg^{-1})(\mathbb{Q})\right|^{-1}$$

for $(P, g) = (P_{12}, 1)$ or (P_{12}, n_{12}) or $(P_2, 1)$. For $(P, g) = (P_{12}, 1)$ or (P_{12}, n_{12}),

$$(-1)^{\dim A_{gMg^{-1}}/A_{M_P}} = (-1)^0 = 1$$

$$(n^{M_P}_{gMg^{-1}})^{-1} = 1.$$

For $(P, g) = (P_2, 1)$

$$(-1)^{\dim A_{gMg^{-1}}/A_{M_P}} = (-1)^1 = -1$$

$$(n^{M_P}_{gMg^{-1}})^{-1} = 2^{-1}\ )$$

Here the non-trivial element in $\mathrm{Nor}_{M_P}(gMg^{-1})(\mathbb{Q})/(gMg^{-1})(\mathbb{Q})$ is represented by n_{12} in the odd case, and by n_{12} times an element of $\mathrm{O}(V_2^{\perp}/V_2)$ of determinant -1 in the even case. □

We next compute the Tr terms in (4.11.1). Note that the case $S = \{2\}$ in the following lemma was already used in the proof of Proposition 4.10.1.

Lemma 4.11.2. *For $S = \{12\}$ or $\{2\}$, we have*

$$\mathrm{Tr}(\gamma, R\Gamma(\mathrm{Lie}(N_S), \mathbb{V})_{>t_S}) = \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ \forall i\in S}} \epsilon(\omega)\Delta_{M_S}(\gamma)^{-1}(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma). \tag{4.11.2}$$

Proof. The lemma follows from Kostant's Theorem similarly as (4.9.5). □

We now break our discussion into the odd case and the even case.

4.11.3 The odd case

Definition 4.11.4. Define

$$\eta_2(\gamma) := \prod_{\alpha\in\Phi^+_{M_2}-\Phi^+_M} \frac{|1-\alpha^{-1}(\gamma)|}{1-\alpha^{-1}(\gamma)}.$$

For $\omega \in \Omega$, define

$$N_1(\omega) = \begin{cases} 1, & \langle\omega(\lambda+\rho), \varpi_i\rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_2(\omega) = \begin{cases} 1, & \langle\omega(\lambda+\rho), \omega_0\varpi_i\rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_3(\omega) = \begin{cases} 1, & \langle\omega(\lambda+\rho), \varpi_2\rangle > 0 \\ 0, & \text{else.} \end{cases}$$

Lemma 4.11.5. *Assume d is odd.*

1. When $0 < b < 1$,

$$L_M(\gamma) \cdot (\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1})^{-1} =$$

$$= \sum_{\omega \in \Omega} [N_1(\omega) - N_2(\omega) + N_3(\omega)] \left[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma) \right].$$

2. When $b > 1$,

$$L_M(\gamma) \cdot (\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1})^{-1} =$$

$$= \sum_{\omega \in \Omega} [N_1(\omega) - N_2(\omega) - N_3(\omega)] \left[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma) \right].$$

3. When $b < 0$,

$$L_M(\gamma) \cdot (\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1})^{-1} =$$

$$= \sum_{\omega \in \Omega} [N_1(\omega) + N_2(\omega) - N_3(\omega)] \left[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma) \right].$$

Proof. Arguing as [60, p. 1701], we have

$$\left|D_M^{M_2}(\gamma)\right|^{1/2} \delta_{P_2(\mathbb{R})}(\gamma)^{1/2}\Delta_{M_2}(\gamma)^{-1} = \eta_2(\gamma)\delta_{P_{12}}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}. \tag{4.11.3}$$

We have $\Phi^+_{M_2} - \Phi^+_M = \{\epsilon_2, \epsilon_2 \pm \epsilon_j, j \geq 3\}$. Note that $\epsilon_2 + \epsilon_j$ is the complex conjugate to $\epsilon_2 - \epsilon_j$, for $j \geq 3$, with respect to the real structure on T_{12}. Hence

$$\eta_2(\gamma) = \frac{\left|1 - \epsilon_2^{-1}(\gamma)\right|}{1 - \epsilon_2^{-1}(\gamma)} = \begin{cases} 1, \ \epsilon_2(\gamma) < 0 \text{ or } > 1 \\ -1, \ 0 < \epsilon_2(\gamma) < 1 \end{cases}.$$

To proceed, we argue as [60, p. 1702].

Denote $\gamma' := n_{12}\gamma n_{12}^{-1}$. The action $\mathrm{Int}(n_{12})$ on T_M is induced by the element $\omega_0 = s_{\epsilon_2}$ (i.e. reflection along ϵ_2) in the Weyl group Ω. For $\omega \in \Omega$, we have:

$$\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma') = \prod_{\alpha\in\Phi^+} |\alpha(\gamma')|^{1/2} \prod_{\alpha\in\Phi^+_M} |\alpha(\gamma')|^{-1/2} \prod_{\alpha\in\Phi^+\cap(-\omega\Phi^+)} \alpha(\gamma')^{-1}$$

$$= \prod_{\alpha\in\omega_0\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi^+_M} |\alpha(\gamma')|^{-1/2} \prod_{\alpha\in\omega_0\Phi^+\cap(-\omega_0\omega\Phi^+)} \alpha(\gamma)^{-1}.$$

Also by definition

$$\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma) \prod_{\alpha\in\Phi(\omega_0\omega)} \alpha^{-1}(\gamma) = \prod_{\alpha\in\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi^+_M} |\alpha(\gamma)|^{-1/2} \prod_{\alpha\in\Phi^+\cap(-\omega_0\omega\Phi^+)} \alpha(\gamma)^{-1}.$$

Since $\gamma(\gamma')^{-1} \in T_{12}$ is in the image of $\epsilon_2^\vee$, which is a central cocharacter of $M = M_{12}$, we know that $\alpha(\gamma) = \alpha(\gamma')$ for each $\alpha \in \Phi_M$. Hence

$$\begin{aligned} \frac{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma')}{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma) \prod_{\alpha\in\Phi(\omega_0\omega)} \alpha^{-1}(\gamma)} &= \frac{\prod_{\alpha\in\omega_0\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\omega_0\Phi^+\cap(-\omega_0\omega\Phi^+)} \alpha(\gamma)^{-1}}{\prod_{\alpha\in\Phi^+} |\alpha(\gamma)|^{1/2} \prod_{\alpha\in\Phi^+\cap(-\omega_0\omega\Phi^+)} \alpha(\gamma)^{-1}} \\ &= \prod_{\alpha\in\Phi^+\cap(-\omega_0\Phi^+)} |\alpha(\gamma)|^{-1} \alpha(\gamma). \end{aligned} \tag{4.11.4}$$

We have $\Delta_M(\gamma) = \Delta_M(\gamma')$, because $\gamma(\gamma')^{-1} \in Z_M$.

Now for $\omega \in \Omega$,

$$(\omega\lambda)(\gamma') = (\omega_0\omega\lambda)(\gamma),$$

$$\epsilon(\omega) = \epsilon(\omega_0)^{-1}\epsilon(\omega_0\omega) = -\epsilon(\omega_0\omega),$$

$$\langle \omega(\lambda+\rho), \varpi_i \rangle = \langle \omega_0\omega(\lambda+\rho), \omega_0\varpi_i \rangle.$$

Hence by (4.11.2),

(4.11.5)

$$\operatorname{Tr}(\gamma', R\Gamma(\operatorname{Lie}(N_{12}), \mathbb{V})_{>t_{12}}) = \sum_{\omega\in\Omega, \langle\omega(\lambda+\rho),\omega_0\varpi_i\rangle>0} -\epsilon(\omega)\Delta_M(\gamma)^{-1}(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega_0\omega)} \alpha^{-1}(\gamma').$$

Combining (4.11.1)(4.11.2) (4.11.3) (4.11.4) (4.11.5), we have

(4.11.6) $L_M(\gamma)\cdot(\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1})^{-1} =$

$$\begin{aligned}
&= \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
&\quad - \prod_{\alpha\in\Phi(\omega_0)} |\alpha(\gamma)|^{-1}\alpha(\gamma) \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\omega_0\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
&\quad - \eta_2(\gamma) \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
&= \sum_{\omega\in\Omega} \left[N_1(\omega) - N_2(\omega) \left(\prod_{\alpha\in\Phi(\omega_0)} |\alpha(\gamma)|^{-1}\alpha(\gamma) \right) - \eta_2(\gamma)N_3(\omega) \right] \times \\
&\quad \times \left[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \right] \\
&\overset{\dagger}{=} \sum_{\omega\in\Omega} [N_1(\omega) - \operatorname{sgn}(\epsilon_2(\gamma))N_2(\omega) - \eta_2(\gamma)N_3(\omega)] \left[\epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \right],
\end{aligned}$$

which implies lemma. Here the equality † follows from the following observation: We have $\Phi(\omega_0) = \{\epsilon_2, \epsilon_2 \pm \epsilon_j,\ j \geq 3\}$, which appeared before as $\Phi^+_{M_2} - \Phi^+_M$. We already observed that $\epsilon_2 + \epsilon_j$ is complex conjugate to $\epsilon_2 - \epsilon_j$, for $j \geq 3$. Hence we have

$$\prod_{\alpha\in\Phi(\omega)} |\alpha(\gamma)|^{-1}\alpha(\gamma) = |\epsilon_2(\gamma)|^{-1}\epsilon_2(\gamma).$$

$\square$

Next we study Φ_M^G and a variant of it, denoted by $\Phi^G_{M,\mathrm{endosc}}$. As usual we have

(4.11.7)

$$\Phi_M^G(\gamma,\Theta)=(-1)^{q(G)}\epsilon_R(\gamma)\delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}\epsilon(\omega)n(\gamma,\omega B)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Definition 4.11.6. When $a,b>0$, consider $R_{\mathrm{endosc}}:=\{\pm\epsilon_1,\pm\epsilon_2\}$. This root system in $X^*(A_{12})_{\mathbb{R}}$ also has -1 in its Weyl group. We use this to define $n_{\mathrm{endosc}}(\gamma,\omega)$. When it is not the case that $a,b>0$, we define $n_{\mathrm{endosc}}=0$.

Definition 4.11.7.

$$\Phi_M^G(\gamma,\Theta)_{\mathrm{endosc}}:=(-1)^{q(G)}\epsilon_R(\gamma)\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\times$$

$$\times\sum_{\omega\in\Omega}\epsilon(\omega)n_{\mathrm{endosc}}(\gamma,\omega B)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Here, as always, $\epsilon_R(\gamma)$ is -1 to the number of positive roots in R that send γ into $]0,1[$. We deliberately use ϵ_R instead of $\epsilon_{R_{\mathrm{endosc}}}$.

Proposition 4.11.8. *When $ab<0$, we have $\Phi_M^G(\gamma,\Theta)=\Phi_M^G(\gamma,\Theta)_{\mathrm{endosc}}=0$. When $ab>0$, we have*

$$4(-1)^{q(G)}L_M(\gamma)=\Phi_M^G(\gamma,\Theta)+\Phi_M^G(\gamma,\Theta)_{\mathrm{endosc}},\ \textit{for } x_1<-|x_2|\,,x_2\neq 0,$$

where we have written $x=(\log|a|\,,\log|b|)=(x_1,x_2)$.

Proof. We have $T_{12}\cong\mathbb{G}_m\times\mathbb{G}_m\times T_{V_2^\perp/V_2}$. Its maximal split sub-torus is $A_{12}=\mathbb{G}_m\times\mathbb{G}_m$. The maximal compact subgroup $T_{12}(\mathbb{R})_1$ of $T_{12}(\mathbb{R})$ is $\{\pm1\}\times\{\pm1\}\times T_{V_2^\perp/V_2}(\mathbb{R})$. Under the decomposition

$$T_{12}(\mathbb{R}) = A_{12}(\mathbb{R})^0 \times T_{12}(\mathbb{R})_1,$$

the coordinate of γ in $A_{12}(\mathbb{R})^0$ is $(|a|, |b|) = \exp(x)$, where we view $x = (x_1, x_2)$ as an element of $\mathbb{R}^2 = \mathrm{Lie}(A_{12}) = X_*(A_{12})_{\mathbb{R}}$.

We first treat the case $ab < 0$. Then $\Phi_M^G(\gamma, \Theta)_{\mathrm{endosc}} = 0$ by definition. To show $\Phi_M^G(\gamma, \Theta) = 0$, note that $R_\gamma = \{\pm\epsilon_1\}$ or $\{\pm\epsilon_2\}$. Thus as a root system in $X^*(A_{12})_{\mathbb{R}} \cong \mathbb{R}^2$, R_γ does not have -1 in its Weyl group. Then as argued in [18, §4], γ cannot lie inside $Z_G(\mathbb{R})\,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R})) = G(\mathbb{R})^0$ (which can also be seen directly), and hence all $n(\gamma, \omega B) = 0$, and $\Phi_M^G(\gamma, \Theta) = 0$ as desired.

We now assume $ab > 0$. We first assume $a, b > 0$. In the range $x_1 < -|x_2|$, there are three cases to consider, namely $0 < a < b < 1$ or $0 < ab < 1 < b$. We have

$$\epsilon_R(\gamma) = \begin{cases} 1, & 0 < a < b < 1 \\ -1, & 0 < ab < 1 < b. \end{cases}$$

Comparing Lemma 4.11.5 and the formulas (cf. (4.11.7) and Definition 4.11.7) for $\Phi_M^G(\gamma, \Theta)$ and $\Phi_M^G(\gamma, \Theta)_{\mathrm{endosc}}$, it suffices to prove that, for any $\omega \in \Omega$:

- When $0 < a < b < 1$,

$$\frac{1}{4}(n(\gamma, \omega B) + n_{\mathrm{endosc}}(\gamma, \omega B)) = N_1(\omega) - N_2(\omega) + N_3(\omega).$$

- When $0 < ab < 1 < b$,

$$\frac{1}{4}(n(\gamma, \omega B) + n_{\mathrm{endosc}}(\gamma, \omega B)) = -N_1(\omega) + N_2(\omega) + N_3(\omega).$$

We call the two sides of the above two desired relations LHS and RHS.

Since obviously $\gamma \in T_{12}(\mathbb{R})^0 \subset G(\mathbb{R})^0$, the numbers $n(\gamma, \omega B)$ are computed using the root system R_γ in $X^*(A_{12})_{\mathbb{R}}$. We also have $R_\gamma = R$. In $\mathbb{R}^2 = \{(x_1, x_2)\}$, the two coordinate axes and the two diagonals $x_1 = \pm x_2$ divide $\mathbb{R}^2$ into eight zones. Call the zone $0 < x_2 < x_1$ zone (I), and number the other zones counterclockwise, with $(II), (III), \cdots, (VIII)$. We also use the name of the zone to denote its characteristic function.

The functions $\omega \mapsto n(\gamma, \omega B), \omega \mapsto n_{\text{endosc}}(\gamma, \omega B), \omega \mapsto N_i(\omega)$, which are functions in ω, only depend on $\wp(\omega(\lambda + \rho)) \in X^*(A_{12})_{\mathbb{R}} \cong \mathbb{R}^2$. We now view them as functions on $\mathbb{R}^2$.

When $0 < a < b < 1$,

$$4^{-1} n(\gamma, \cdot) = (II) + (VIII)$$

$$4^{-1} n_{\text{endosc}}(\gamma, \cdot) = (I) + (II)$$

so LHS $= (I) + 2(II) + (VIII)$.

$$N_1 = (I) + (II) + (VIII),$$

$$N_2 = (I) + (VII) + (VIII),$$

$$N_3 = (I) + (II) + (VII) + (VIII)$$

so RHS $= (I) + 2(II) + (VIII)$. We have LHS = RHS as desired.

When $0 < ab < 1 < b$,

$$4^{-1} n(\gamma, \cdot) = (I) + (VII)$$

$$4^{-1} n_{\text{endosc}}(\gamma, \cdot) = (VII) + (VIII)$$

so LHS $= (I) + 2(VII) + (VIII)$. The computations for N_1, N_2, N_3 remain the same, so RHS $= (I) + 2(VII) + (VIII)$. We have LHS = RHS as desired.

We now treat the case $a, b < 0$. In this case $\Phi^G_M(\gamma, \Theta)_{\text{endosc}} = 0$ by definition. We have

$\epsilon_R(\gamma) = 1$. Comparing Lemma 4.11.5 and (4.11.7), it suffices to prove that, for any $\omega \in \Omega$:

$$\frac{1}{4} n(\gamma, \omega B) = N_1(\omega) + N_2(\omega) - N_3(\omega).$$

We claim that $\gamma \in G(\mathbb{R})^0$. In fact, since $T_{V_2^\perp/V^2}(\mathbb{R})$ is connected (being a product of copies of the circle group), it suffices to prove that

$$\gamma_1 := (-1, -1, 1) \in \mathbb{G}_m(\mathbb{R}) \times \mathbb{G}_m(\mathbb{R}) \times T_{V_2^\perp/V_2}(\mathbb{R})$$

is in $G(\mathbb{R})^0$. But γ_1 acts as -1 on $\mathbb{R}X_1 + \mathbb{R}X_2$ and on $\mathbb{R}Y_1 + \mathbb{R}Y_2$, where $X_i = e_i + e_i'$ and $Y_i = e_i - e_i'$. Now $\mathbb{R}X_1 + \mathbb{R}X_2$ is a positive definite plane and $\mathbb{R}Y_1 + \mathbb{R}Y_2$ is a negative definite plane, and γ_1 acts on both of them with determinant 1. It follows that $\gamma, \gamma_1 \in G(\mathbb{R})^0$.

By the claim, the numbers $n(\gamma, \omega B)$ are computed using the root system R_γ in $X^*(A_{12})_{\mathbb{R}} = \mathbb{R}^2$. We have $R_\gamma = \{\pm\epsilon_1 \pm \epsilon_2\}$.

Since $x_1 < -|x_2|$,

$$4^{-1} n(\gamma, \cdot) = (I) + (VIII)$$

Now

$$N_1 = (I) + (II) + (VIII),$$

$$N_2 = (I) + (VII) + (VIII),$$

$$N_3 = (I) + (II) + (VII) + (VIII),$$

so $N_1 + N_2 - N_3 = (I) + (VIII) = 4^{-1} n(\gamma, \cdot)$, as desired.

□

The following two propositions will also be needed later.

Proposition 4.11.9. *Assume d is odd. Let $\gamma = (a, b, \gamma_{V_2^\perp/V_2}) \in T_{12}(\mathbb{R})^0$, with $ab > 0, a \neq b$.*

Let $\gamma' \in T_{12}(\mathbb{R})^0$ be the element obtained from γ by switching a and b. Then

$$\Phi^G_M(\gamma, \Theta) = \Phi^G_M(\gamma', \Theta)$$

and

$$\epsilon_R(\gamma)\epsilon_{R_{\mathrm{endosc}}}(\gamma)\Phi^G_M(\gamma, \Theta)_{\mathrm{endosc}} = -\epsilon_R(\gamma')\epsilon_{R_{\mathrm{endosc}}}(\gamma')\Phi^G_M(\gamma', \Theta)_{\mathrm{endosc}}.$$

Here $\epsilon_{R_{\mathrm{endosc}}}(\gamma)$ is -1 to the number of positive roots in R_{endosc} that send γ into $]0, 1[$.

Proof. The first equality is true because $\gamma \mapsto \gamma'$ is induced by an element of $(\mathrm{Nor}_G\, M_{12})(\mathbb{R})$, and $\Phi^G_M(\cdot, \Theta)$ is invariant under the latter group. In the following we prove the second equality. The functions $\epsilon_{R_{\mathrm{endosc}}}$ and Δ_M are invariant under $\gamma \mapsto \gamma'$, so we need to check

$$\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2} \sum_{\omega} \epsilon(\omega) n_{\mathrm{endosc}}(\gamma, \omega B)(\omega\lambda)(\gamma) \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma)$$

$$= -\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2} \sum_{\omega} \epsilon(\omega) n_{\mathrm{endosc}}(\gamma', \omega B)(\omega\lambda)(\gamma') \prod_{\alpha \in \Phi(\omega)} \alpha^{-1}(\gamma').$$

Let $s := s_{\epsilon_1 - \epsilon_2} \in \Omega$, so that $\gamma' = s\gamma$. For any $\omega \in \Omega$, we have $n_{\mathrm{endosc}}(\gamma', s\omega B) = n_{\mathrm{endosc}}(\gamma, \omega B)$. (Both are zero in case $a, b < 0$.) Arguing in the same way as the discussion around (4.11.4) (4.11.5), we reduce to checking the following:

$$-1 = -\prod_{\alpha \in \Phi^+ \cap (-s\Phi^+)} |\alpha(\gamma)|^{-1} \alpha(\gamma).$$

This is true because $\Phi^+ \cap (-s\Phi^+) = \{\epsilon_1 - \epsilon_2\}$.

□

Proposition 4.11.10. *Assume d is odd. Let $\gamma = (a, b, \gamma_{V_2^\perp / V_2}) \in T_{12}(\mathbb{R})^0$, with $ab > 0, a \neq b$.*

Let $\gamma' \in T_{12}(\mathbb{R})^0$ be the element obtained from γ by switching replacing a with a^{-1}. Then

$$\Phi_M^G(\gamma, \Theta) = \Phi_M^G(\gamma', \Theta),$$

and

$$\epsilon_R(\gamma)\epsilon_{R_{\mathrm{endosc}}}(\gamma)\Phi_M^G(\gamma, \Theta)_{\mathrm{endosc}} = \epsilon_R(\gamma')\epsilon_{R_{\mathrm{endosc}}}(\gamma')\Phi_M^G(\gamma', \Theta)_{\mathrm{endosc}}.$$

Proof. As in the proof of Proposition 4.11.9, the first equality is true because $\gamma \mapsto \gamma'$ is induced by an element of $(\mathrm{Nor}_G M_{12})(\mathbb{R})$. To prove the second equality, let $s := s_{\epsilon_1} \in \Omega$, so that $\gamma' = s\gamma$. Arguing similarly as in the proof of Proposition 4.11.9, we need only check that

$$\prod_{\alpha \in \Phi^+ \cap (-s\Phi^+)} |\alpha(\gamma)|^{-1} \alpha(\gamma) = \begin{cases} 1, & \epsilon_{R_{\mathrm{endosc}}}(s\gamma)\epsilon_{R_{\mathrm{endosc}}}(\gamma) = -1 \\ -1, & \epsilon_{R_{\mathrm{endosc}}}(s\gamma)\epsilon_{R_{\mathrm{endosc}}}(\gamma) = 1 \end{cases},$$

or equivalently,

$$\prod_{\alpha \in \Phi^+ \cap (-s\Phi^+)} |\alpha(\gamma)|^{-1} \alpha(\gamma) = \mathrm{sgn}(a). \tag{4.11.8}$$

Now to compute the LHS of (4.11.8), we may restrict the product to only those α that are real, because $\Phi^+ \cap (-s\Phi^+)$ is stable under complex conjugation and the product over all the non-real roots in it is obviously 1. The real roots in $\Phi^+ \cap (-s\Phi^+)$ are $\epsilon_1, \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2$. Hence the LHS of (4.11.8) is equal to $\mathrm{sgn}(a)^3 = \mathrm{sgn}(a)$, as desired. □

4.11.11 The even case

Definition 4.11.12. Denote by

$$\mathrm{a} : G \to G$$

the (non-inner) automorphism $\operatorname{Int} n_{12}$ of G, where $n_{12} \in \mathrm{O}(V)(\mathbb{Q})$ is defined in Definition 2.3.2. Let ${}^{\mathrm{a}}\mathbb{V}$ be the representation of $G_{\mathbb{C}}$ obtained from $\mathbb{V}$ composed with a. Define ${}^{\mathrm{a}}L_M(\gamma)$ in the same way as $L_M(\gamma)$, but with ${}^{\mathrm{a}}\mathbb{V}$ in place of $\mathbb{V}$. Define $\lambda' \in X^*(T_{12})$ to be the highest weight of ${}^{\mathrm{a}}\mathbb{V}$.

Definition 4.11.13. Define

$$\eta_2(\gamma) := \prod_{\alpha \in \Phi^+_{M_2} - \Phi^+_M} \frac{|1 - \alpha^{-1}(\gamma)|}{1 - \alpha^{-1}(\gamma)}.$$

For $\omega \in \Omega$, define

$$N_1(\omega) = \begin{cases} 1, & \langle \omega(\lambda + \rho), \varpi_i \rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_2(\omega) = \begin{cases} 1, & \langle \omega(\lambda + \rho), a\varpi_i \rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_3(\omega) = \begin{cases} 1, & \langle \omega(\lambda + \rho), \varpi_2 \rangle > 0 \\ 0, & \text{else} \end{cases}$$

$$N_1'(\omega) = \begin{cases} 1, & \langle \omega(\lambda' + \rho), \varpi_i \rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_2'(\omega) = \begin{cases} 1, & \langle \omega(\lambda' + \rho), \mathrm{a}\varpi_i \rangle > 0,\ i = 1, 2 \\ 0, & \text{else} \end{cases}$$

$$N_3'(\omega) = \begin{cases} 1, & \langle \omega(\lambda' + \rho), \varpi_2 \rangle > 0 \\ 0, & \text{else.} \end{cases}$$

Lemma 4.11.14. *Assume d is even. We have*

$$L_M(\gamma) + {}^{\mathrm{a}}L_M(\gamma) = (\mathscr{A}) + (\mathscr{B}),$$

where

$$(\mathscr{A}) = \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}\left[N_1(\omega) + N_2(\omega) - N_3(\omega)\right]\epsilon(\omega)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma)$$

$$(\mathscr{B}) = \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}\left[N_1'(\omega) + N_2'(\omega) - N_3'(\omega)\right]\epsilon(\omega)(\omega\lambda')(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Proof. As in the odd case, we have

$$\left|D_M^{M_2}(\gamma)\right|^{1/2}\delta_{P_2(\mathbb{R})}(\gamma)^{1/2}\Delta_{M_2}(\gamma)^{-1} = \eta_2(\gamma)\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}. \tag{4.11.9}$$

We have $\Phi^+_{M_2} - \Phi^+_M = \{\epsilon_2 \pm \epsilon_j, j \geq 3\}$. Note that $\epsilon_2 + \epsilon_j$ is the complex conjugate to $\epsilon_2 - \epsilon_j$, for $j \geq 3$, with respect to the real structure on T_{12}. Hence

$$\eta_2(\gamma) = 1.$$

Write $\gamma' := \mathrm{a}(\gamma)$. Combining (4.11.1)(4.11.2) (4.11.9), we have

$$
\begin{aligned}
(4.11.10)\qquad & L_M(\gamma) = \\
& \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\omega\in\Omega, \langle \omega(\lambda+\rho),\varpi_i\rangle>0, i=1,2} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma) \\
& + \delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2}\Delta_M(\gamma')^{-1} \sum_{\omega\in\Omega, \langle \omega(\lambda+\rho),\varpi_i\rangle>0, i=1,2} \epsilon(\omega)(\omega\lambda)(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma') \\
& - \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\omega\in\Omega, \langle \omega(\lambda+\rho),\varpi_2\rangle>0} \epsilon(\omega)(\omega\lambda)(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha^{-1}(\gamma).
\end{aligned}
$$

The automorphism a preserves T_{12}, and acts on $X^*(T_{12})$ as s_{ϵ_2}, i.e. the reflection along ϵ_2. Of course this operator is not an element of Ω. Let ω_0 be an inner automorphism of $G_{\mathbb{C}}$ which normalizes T and maps the positive system Φ^+ to the positive system corresponding to the following root basis:

$$\left\{\epsilon_m - \epsilon_{m-1}, \epsilon_{m-1} - \epsilon_{m-2}, \cdots, \epsilon_5 - \epsilon_4, \epsilon_4 - \epsilon_3, \epsilon_3 - \epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\right\}.$$

Now a preserves $\omega_0\Phi^+$. The highest weight of $\mathbb{V}$ with respect to $\omega_0\Phi^+$ is $\omega_0\lambda$. Hence the highest weight λ' of ${}^{\mathrm{a}}\mathbb{V}$ with respect to $\omega_0\Phi^+$ is $\mathrm{a}\omega_0\lambda$. Consequently, λ', the highest weight of ${}^{\mathrm{a}}\mathbb{V}$ with respect to Φ^+, is given by

$$\lambda' = \mathrm{a}'\lambda,$$

where

$$\mathrm{a}' := \omega_0^{-1}\mathrm{a}\omega_0.$$

Note that a' is an involution of G preserving Φ^+. Moreover, the map

$$\mathrm{Aut}(X^*(T_{12})) \to \mathrm{Aut}(X^*(T_{12}))$$

$$\omega \mapsto \mathrm{a} \circ \omega \circ \mathrm{a}$$

preserves the subset $\Omega \subset \mathrm{Aut}(X^*(T_{12}))$. Since ω_0 is inner, it follows that the following map also preserves Ω:

$$\mathrm{Aut}(X^*(T_{12})) \to \mathrm{Aut}(X^*(T_{12}))$$

$$\omega \mapsto \mathrm{a} \circ \omega \circ \mathrm{a}'.$$

Applying (4.11.10) to ${}^{\mathrm{a}}\mathbb{V}$ and adding the resulting equation to the original one, we get

$$L_M(\gamma) + {}^{\mathrm{a}}L_M(\gamma) = (A) + (B) + (C), \tag{4.11.11}$$

where

$$\begin{aligned}
&(A) = \\
&\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega(\lambda))(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma)^{-1} \\
&+ \delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2}\Delta_M(\gamma')^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda'+\rho),\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega(\lambda'))(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma')^{-1}
\end{aligned}$$

$$\begin{aligned}
&(B) = \\
&\delta_{P_{12}(\mathbb{R})}(\gamma')^{1/2}\Delta_M(\gamma')^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega(\lambda))(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma')^{-1} \\
&+ \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda'+\rho),\varpi_i\rangle>0,\ i=1,2}} \epsilon(\omega)(\omega(\lambda'))(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma)^{-1}
\end{aligned}$$

$$(C) =$$

$$- \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega(\lambda))(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma)^{-1}$$

$$- \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1} \sum_{\substack{\omega\in\Omega \\ \langle\omega(\lambda'+\rho),\varpi_2\rangle>0}} \epsilon(\omega)(\omega(\lambda'))(\gamma) \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma)^{-1}.$$

We first compute (A). We have

$$(\omega(\lambda'))(\gamma') = ((\mathrm{a}\omega\mathrm{a}')\lambda)(\gamma),$$

where as we have observed, $\mathrm{a}\omega\mathrm{a}' \in \Phi$.

We would like to make the change of variable $\omega \mapsto \mathrm{a}\omega\mathrm{a}'$ in the second summation in the definition of (A). We have

$$\langle \mathrm{a}\omega\mathrm{a}'(\mathrm{a}'\lambda + \rho), \varpi_i\rangle = \langle \mathrm{a}\omega(\lambda + \rho), \varpi_i\rangle = \langle \omega(\lambda + \rho), \mathrm{a}\varpi_i\rangle$$

$$\epsilon(\mathrm{a}\omega\mathrm{a}') = \epsilon(\omega).$$

Hence

$$\sum_{\omega\in\Omega, \langle\omega(\lambda'+\rho),\varpi_i\rangle>0, i=1,2} \epsilon(\omega)(\omega(\lambda'))(\gamma') \prod_{\alpha\in\Phi(\omega)} \alpha(\gamma)^{-1} =$$

$$= \sum_{\omega\in\Omega, \langle\omega(\lambda+\rho),\mathrm{a}\varpi_i\rangle>0, i=1,2} \epsilon(\omega)(\omega(\lambda))(\gamma) \prod_{\alpha\in\Phi(\mathrm{a}\omega\mathrm{a}')} \alpha(\gamma')^{-1}.$$

Now

$$\prod_{\alpha\in\Phi(\mathrm{a}\omega\mathrm{a}')} \alpha(\gamma')^{-1} = \prod_{\alpha\in\Phi^+\cap(-\mathrm{a}\omega\mathrm{a}'\Phi^+)} \alpha(\gamma')^{-1} = \prod_{\alpha\in\mathrm{a}\Phi^+\cap(-\omega\mathrm{a}'\Phi^+)} \alpha(\gamma)^{-1} = \prod_{\alpha\in\mathrm{a}\Phi^+\cap(-\omega\Phi^+)} \alpha(\gamma)^{-1},$$

where the last equality is because $\mathrm{a}'\Phi^+ = \Phi^+$. Therefore,

$$\text{(4.11.12)} \qquad \frac{\prod_{\alpha\in\Phi(\mathrm{a}\omega\mathrm{a}')}\alpha(\gamma')^{-1}}{\prod_{\alpha\in\Phi(\omega)}\alpha(\gamma)^{-1}} = \frac{\prod_{\alpha\in\mathrm{a}\Phi^+\cap(-\omega\Phi^+)}\alpha(\gamma)^{-1}}{\prod_{\alpha\in\Phi^+\cap(-\omega\Phi^+)}\alpha(\gamma)^{-1}} = \prod_{\alpha\in\Phi^+\cap(-\mathrm{a}\Phi^+)}\alpha(\gamma)^{+1}.$$

On the other hand,

$$\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma) = \prod_{\alpha\in\Phi^+-\Phi^+_M}|\alpha(\gamma)|^{1/2},$$

so

$$\text{(4.11.13)} \qquad \frac{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma')}{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma)} = \frac{|\epsilon_2(\gamma')|^{m-2}}{|\epsilon_2(\gamma)|^{m-2}} = |\epsilon_2(\gamma)|^{4-2m} = \left|\prod_{\alpha\in\Phi^+\cap(-\mathrm{a}\Phi^+)}\alpha(\gamma)\right|^{-1}.$$

Now we have

$$\Phi^+\cap(-\mathrm{a}\Phi^+) = \{\epsilon_2 \pm \epsilon_j \ j\geq 3\}$$

and

$$\Phi^+ - \Phi^+_M = \{\epsilon_1\pm\epsilon_2, \epsilon_1\pm\epsilon_j, \epsilon_2\pm\epsilon_j, \ j\geq 3\}.$$

Therefore combining (4.11.12) and (4.11.13) we have

$$\frac{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma')\prod_{\alpha\in\Phi(\mathrm{a}\omega\mathrm{a}')}\alpha(\gamma')^{-1}}{\delta^{1/2}_{P_{12}(\mathbb{R})}(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha(\gamma)^{-1}} = \prod_{\alpha\in\{\epsilon_2\pm\epsilon_j, \ j\geq 3\}}\alpha(\gamma)\,|\alpha(\gamma)|^{-1} = 1,$$

where the last equality follows from the observation that $\epsilon_2+\epsilon_j$ is complex conjugate to $\epsilon_2-\epsilon_j$, for $j\geq 3$.

Finally, $\Delta_M(\gamma) = \Delta_M(\gamma')$, by definition of Δ_M.

We conclude that

$$\text{(4.11.14)} \qquad (A) = \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}[N_1(\omega)+N_2(\omega)]\,\epsilon(\omega)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Observe that (B) is obtained from (A) by interchanging the roles of λ and λ'. Thus we get

$$(4.11.15) \qquad (B) = \delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}\left[N_1'(\omega)+N_2'(\omega)\right]\epsilon(\omega)(\omega\lambda')(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).$$

Finally, by definition

$$(4.11.16) \qquad \begin{aligned}(C) = &-\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}N_3(\omega)\epsilon(\omega)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma)\\ &-\delta_{P_{12}(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}N_3'(\omega)\epsilon(\omega)(\omega\lambda')(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma).\end{aligned}$$

The lemma follows from (4.11.11) (4.11.14) (4.11.15) (4.11.16). □

Proposition 4.11.15. *Assume d is even. Let Θ' be the analogue of Θ, obtained by replacing $\mathbb{V}$ with ${}^{\mathrm{a}}\mathbb{V}$. When $ab<0$, we have*

$$\Phi_M^G(\gamma,\Theta)=\Phi_M^G(\gamma,\Theta')=0.$$

When $ab>0$, we have

$$4(-1)^{q(G)}(L_M(\gamma)+{}^{\mathrm{a}}L_M(\gamma))=\Phi_M^G(\gamma,\Theta)+\Phi_M^G(\gamma,\Theta'),\ \textit{for } x_1<-\left|x_2\right|, x_2\neq 0,$$

where we have written $x=(\log|a|,\log|b|)=(x_1,x_2)$.

Proof. As in the odd case, we have

$$\Phi_M^G(\gamma,\Theta)=(-1)^{q(G)}\epsilon_R(\gamma)\delta_{P_1(\mathbb{R})}(\gamma)^{1/2}\Delta_M(\gamma)^{-1}\sum_{\omega\in\Omega}\epsilon(\omega)n(\gamma,\omega B)(\omega\lambda)(\gamma)\prod_{\alpha\in\Phi(\omega)}\alpha^{-1}(\gamma),$$

where $\epsilon_R(\gamma)$ is -1 to the number of positive roots in R that send γ into $]0,1[$.

Now if $ab < 0$, then $R_\gamma = \emptyset$. Thus as a root system in $X^*(A_{12})_{\mathbb{R}} \cong \mathbb{R}^2$, R_γ does not have -1 in its Weyl group. Then as argued in [18, §4] , γ cannot lie inside $Z_G(\mathbb{R})\,\mathrm{im}(G^{\mathrm{SC}}(\mathbb{R})) = G(\mathbb{R})^0$, and hence all $n(\gamma, \omega B) = 0$. Therefore $\Phi_M^G(\gamma, \Theta) = 0$, and for the same reason $\Phi_M^G(\gamma, \Theta') = 0$.

Suppose $ab > 0$. Firstly note that in the range $x_1 < -|x_2|$, we have $\epsilon_R(\gamma) = 1$. In view of Lemma 4.11.14, it suffices to prove that, for any $\omega \in \Omega$:

- $4^{-1} n(\gamma, \omega B) = N_1(\omega) + N_2(\omega) - N_3(\omega)$.
- $4^{-1} n'(\gamma, \omega B) = N_1'(\omega) + N_2'(\omega) - N_3'(\omega)$.

The proofs of the two statements are of course the same. We prove the first statement in the following.

Arguing as in the odd case, we know that $\gamma \in G(\mathbb{R})^0$, so the numbers $n(\gamma, \omega B)$ are computed using the root system R_γ in $X^*(A_{12})_{\mathbb{R}}$. We have $R_\gamma = R = \{\pm\epsilon_1 \pm \epsilon_2\}$.

Since $x_1 < -|x_2|$,

$$4^{-1} n(\gamma, \cdot) = (I) + (VIII)$$

Now

$$N_1 = (I) + (II) + (VIII),$$

$$N_2 = (I) + (VII) + (VIII),$$

$$N_3 = (I) + (II) + (VII) + (VIII).$$

Hence $N_1 + N_2 - N_3 = (I) + (VIII) = 4^{-1} n(\gamma, \cdot)$, as desired. $\square$

5 Global and v-adic endoscopy

5.1 Global and v-adic endoscopic data

We can fix representatives for the elliptic endoscopic data for $G_{\mathbb{Q}}$ or $G_{\mathbb{Q}_v}$, v a finite place, just as for $G_{\mathbb{R}}$. More precisely, we will always use the quasi-split quadratic space with the same dimension and discriminant as (V, q) (over $\mathbb{Q}$ or $\mathbb{Q}_v$) to represent the quasi-split inner twisting of G; and we will use quasi-split quadratic spaces with *trivial discriminant* to represent the endoscopic groups in the *odd case*, and use quasi-split quadratic spaces, whose *discriminants multiply to the discriminant of* (V, q), to represent the endoscopic groups in *the even case*. Here, quasi-split quadratic spaces with given discriminants are defined in Definition 3.1.1 and Definition 3.2.1. Moreover, in the even case, zero-dimensional quadratic spaces with non-trivial discriminants and two-dimensional quadratic spaces with trivial discriminants will *not* appear in the endoscopic groups. As in the case over $\mathbb{R}$, for all the elliptic endoscopic data over $\mathbb{Q}$ or $\mathbb{Q}_v$ we can and do choose a fixed L embedding from the L-group of the endoscopic group to ${}^LG$. In the odd case this is trivial, since all the endoscopic groups are split. In the even case this is to be specified below.

Thus we may use pairs (d^+, d^-) (resp. quadruples $(d^+, \delta^+, d^-, \delta^-)$) to index elliptic endoscopic data in the odd (resp. even) case, just as over $\mathbb{R}$. Of course now in the even case, $\delta^{\pm}$ lies in $\mathbb{Q}^\times/\mathbb{Q}^{\times,2}$ or $\mathbb{Q}_v^\times/\mathbb{Q}_v^{\times,2}$, rather than $\mathbb{R}^\times/\mathbb{R}^{\times,2} \cong \{\pm 1\}$. Also, for $M \in \{M_1, M_2, M_{12}\}$ a standard Levi subgroup of G, the (bi-elliptic) G-endoscopic data for M will be represented similarly as over $\mathbb{R}$. They are parametrized by the parameters (A, d^+, d^-) in the odd case and $(A, d^+, \delta^+, d^-, \delta^-)$ in the even case in exactly the same way as over $\mathbb{R}$, the only difference being that now in the even case δ^+ and δ^- range over $\mathbb{Q}^\times/\mathbb{Q}^{\times,2}$ or $\mathbb{Q}_v^\times/\mathbb{Q}_v^{\times,2}$.

We use notations $\mathcal{E}(G_F), \mathcal{E}(M_F), \mathcal{E}_{G_F}(M_F)$ analogously to the case over $\mathbb{R}$, where $F = \mathbb{Q}$ or $\mathbb{Q}_v$. When the context is clear, we omit the subscript F.

We make explicit some details in the even case. Let $F = \mathbb{Q}$ or $\mathbb{Q}_v$ where v is an odd prime.

5.1.1 Even case with $\delta \notin F^{\times,2}$.

In this case G^* is quasi-split but not split. Denote $F_\delta := F(\sqrt{\delta})$. We represent ${}^LG$ as $\hat{G} \rtimes \mathrm{Gal}(F_\delta/F)$. Here $\hat{G} = \mathrm{SO}_d(\mathbb{C})$ as in the real case, and the non-trivial element in $\mathrm{Gal}(F_\delta/F)$ acts on $\hat{G}$ by conjugation by the permutation matrix on $\mathbb{C}^d$ that switches $\hat{e}_{d/2}$ and $\hat{e}_{d/2+1}$.

Consider an endoscopic group $H = \mathrm{SO}(d^+, \delta^+) \times \mathrm{SO}(d^-, \delta^-)$. Here $d^+ + d^- = d$ and $\delta^+\delta^- = \delta$. Denote $F_{\delta^\pm} := F(\sqrt{\delta^\pm})$, and $F_H := F(\sqrt{\delta^+}, \sqrt{\delta^-})$.

When $F = \mathbb{Q}$, we know that either F_H is a $\mathbb{Z}/2 \times \mathbb{Z}/2$ extension of F, in which case $F_{\delta^+}, F_{\delta^-}, F_\delta$ are the three quadratic fields inside it, or $F_H = F_\delta$, in which case exactly one of $\delta^\pm$ is in $F^{\times,2}$.

When $F = \mathbb{Q}_v$ $(v > 2)$ and when H is unramified over F, we know that up to $F^{\times,2}$ all of $\delta^+, \delta^-, \delta$ lie in $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,2} \cong \mathbb{Z}/2\mathbb{Z}$. Thus exactly one of $\delta^\pm$ is in $F^{\times,2}$, and we may assume that the other is equal to δ. In this case we have $F_H = F_\delta$. Note that the non-trivial element in $\mathrm{Gal}(F_H/F)$ is the Frobenius.

In the situations of the above two paragraphs, we represent ${}^LH$ as $\hat{H} \rtimes \mathrm{Gal}(F_H/F)$, where we let $\mathrm{Gal}(F_H/F)$ act on the two factors $\widehat{H^+}$ and $\widehat{H^-}$ through its quotients $\mathrm{Gal}(F_H/F_{\delta^-})$ and $\mathrm{Gal}(F_H/F_{\delta^+})$ respectively.

We fix the embedding

$$\eta : {}^LH = \hat{H} \rtimes \mathrm{Gal}(F_H/F) \to {}^LG = \hat{G} \rtimes \mathrm{Gal}(F_\delta/F)$$

similarly to the real case. Here the map $\mathrm{Gal}(F_H/F) \to \hat{G} \rtimes \mathrm{Gal}(F_\delta/F)$ takes τ to $(\rho(\tau), \tau_{F_\delta})$, where $\rho(\tau) \in \hat{G}$ is defined to be

$$\rho(\tau) := \begin{cases} 1, \ \tau|_{F_{\delta^-}} = \mathrm{id}, \\ \\ S, \ \text{otherwise} \end{cases}$$

Here

$$S = \begin{cases} \mathrm{id}, \ d^+ = 0 \\ \\ \text{the permutation matrix } \hat{e}_{m^-} \leftrightarrow \hat{e}_{d-m^-+1} \text{ and } \hat{e}_m \leftrightarrow \hat{e}_{m+1}, \ d^= \neq 0. \end{cases} \tag{5.1.1}$$

similarly to the real case.

Note that when $F = \mathbb{Q}_v$ with $v > 2$ a finite prime, and when H is unramified, we know that ρ takes the Frobenius to the non-trivial element S if and only if $\delta^+ \in \mathbb{Q}_p^{\times,2}, \delta^- = \delta \notin \mathbb{Q}_p^{\times,2}$. We find that in this case, for $t_{\hat{H}} = (t_1, \cdots, t_{m^+}, s_1, \cdots, s_{m^-}) \in \mathcal{T}_{\hat{H}}$ in the diagonal torus of $\hat{H}$, the embedding η takes $t_{\hat{H}} \rtimes \sigma$ not into $\mathcal{T} \rtimes \sigma$, where $\mathcal{T}$ is the diagonal torus of $\hat{G}$, but to an element in ${}^LG$ that is conjugate in ${}^LG$ (for instance, by the permutation matrix that switches $\hat{e}_{m^-}$ and $\hat{e}_m$, and switches $\hat{e}_{d-m^-+1}$ and $(-1)^{m^-+m}\hat{e}_{m^++1}$) to the following element in $\mathcal{T} \rtimes \sigma$:

$$(s_1, \cdots, s_{m^--1}, t_{m^+}, t_1, \cdots, t_{m^+-1}, s_{m^-}) \rtimes \sigma.$$

5.1.2 Even case with $\delta \in F^{\times,2}$

.

In this case G^* is split, and we represent ${}^LG$ as $\hat{G}$. Take an endoscopic group $H = \mathrm{SO}(d^+, \delta^+) \times \mathrm{SO}(d^-, \delta^-)$. We have $\delta^+ = \delta^- \mod F^{\times,2}$. When $\delta^+, \delta^- \in F^{\times,2}$, we know that H is split, and we represent ${}^LH$ as $\hat{H}$. We choose the embedding $\hat{H} \to \hat{G}$ as in the real case.

When $\delta^+, \delta^- \notin F^{\times,2}$, we represent ${}^LH$ as $\hat{H} \rtimes \mathrm{Gal}(F_H/F)$, where $F_H := F(\sqrt{\delta^+}) = F(\sqrt{\delta^-})$. We fix ${}^LH \to {}^LG$ by sending the non-trivial element $\tau \in \mathrm{Gal}(F_H.F)$ to $(S, 1)$, where S is as in (5.1.1). Note that $d^+ = 0$ cannot happen, as we do not allow $d^+ = 0$ and $\delta^+ \neq 1$. We find that in this case, for $t_{\hat{H}} = (t_1, \cdots, t_{m^+}, s_1, \cdots, s_{m^-}) \in \mathcal{T}_{\hat{H}}$ in the diagonal torus of $\hat{H}$, the embedding η takes $t_{\hat{H}} \rtimes \sigma$ not into $\mathcal{T} \rtimes \sigma$, but to an element in ${}^L G$ that is conjugate in ${}^L G$ to the following element in $\mathcal{T} \rtimes \sigma$:

$$(s_1, \cdots, s_{m^- -1}, t_{m^+}, t_1, \cdots, t_{m^+ -1}, s_{m^-}) \rtimes \sigma.$$

5.2 Recall of the Satake isomorphism

We recall the Satake isomorphism, following [9], [7], [19], [71]. Let G_1 be an unramified reductive group over a finite extension F of $\mathbb{Q}_p$. Let q be the residue cardinality of F and let ϖ_F be a uniformizer of F. Let K_1 be a hyperspecial subgroup of $G_1(F)$ determined by a hyperspecial vertex v_0 in the building of G_1. Let S be a maximal split torus of G_1 whose apartment contains v_0, and let T be the centralizer of S. Then T is a maximal torus and minimal Levi of G_1. Let $\Omega(F)$ (resp. Ω) be the relative (resp. absolute) Weyl group of S (resp. T) in G. Then there is a $\mathrm{Gal}(\bar{F}/F)$ action on Ω such that the subgroup of fixed points is naturally identified with $\Omega(F)$. For any choice of Borel subgroup B of G_1 containing T, the Satake isomorphism can be viewed as the following algebra isomorphism, which depends only on S (or T) and not on B:

$$\mathcal{S}_S^{G_1} : \mathcal{H}(G_1(F)//K_1) \xrightarrow{\sim} \mathcal{H}(T(F)/T(F) \cap K_1)^{\Omega(F)} \tag{5.2.1}$$

$$f \mapsto f_T, \ f_T(t) = \delta_{B(F)}^{-1/2}(t) \int_{N_B(F)} f(nt)dn, \ t \in T(F),$$

where N_B is the unipotent radical of B, and we normalize the Haar measure dn on $N_B(F)$ such that $N_B(F) \cap K_1$ has volume 1.

Note that since $T(F) \cap K_1$ is the maximal compact open subgroup of $T(F)$, so the algebra $\mathcal{H}(T(F)/T(F) \cap K_1)$ is independent of the choice of K_1, (as long as K_1 is determined by a hyperspecial vertex in the apartment of S). In fact, we have canonical isomorphisms

$$\mathcal{H}(T(F)/T(F)\cap K_1)\cong\mathcal{H}(S(F)/S(F)\cap K_1)\cong\mathbb{C}[X_*(S)].$$

Thus we will also view the Satake isomorphism (5.2.1) as an isomorphism

$$\mathcal{S}_S^{G_1}:\mathcal{H}(G_1(F)//K_1)\xrightarrow{\sim}\mathbb{C}[X_*(S)]^{\Omega(F)}. \tag{5.2.2}$$

Moreover, note that for any two different maximal split tori S,S' in G_1, there is a canonical isomorphism

$$\mathbb{C}[X_*(S)]^{\Omega(F)}\xrightarrow{\sim}\mathbb{C}[X^*(S')]^{\Omega(F)}$$

induced by conjugation by any element of $G_1(F)$ that sends S to S'.

Definition 5.2.1. Define $\mathscr{A}_{G_1}$ to be the inverse limit of the algebras $\mathbb{C}[X_*(S)]^{\Omega_F}$, for S running through the maximal split tori of G_1, with respect to the canonical isomorphisms between them as transition maps.

The Satake isomorphisms (5.2.2) for various S (whose apartments contain v_0) lift to an isomorphism

$$\mathcal{S}^{G_1}:\mathcal{H}(G_1(F)//K_1)\xrightarrow{\sim}\mathscr{A}_{G_1}, \tag{5.2.3}$$

which depends only on K_1 and no other choices. From (5.2.3) we get a canonical isomorphism between $\mathcal{H}(G_1(F)//K_1)$ and $\mathcal{H}(G_1(F)//K_2)$ for any two hyperspecial subgroups K_1,K_2.

Definition 5.2.2. Denote by $\mathcal{H}^{\mathrm{ur}}(G_1(F))$ the inverse limit of $\mathcal{H}(G_1(F)//K_1)$ over all hyperspecial subgroups K_1 with respect to the canonical isomorphisms between them as transition maps. We call $\mathcal{H}^{\mathrm{ur}}(G_1(F))$ *the canonical unramified Hecke algebra.*

Hence the Satake isomorphism may be viewed as a canonical isomorphism

$$\mathcal{S}^{G_1} : \mathcal{H}^{\mathrm{ur}}(G_1(F)) \xrightarrow{\sim} \mathscr{A}_{G_1}. \tag{5.2.4}$$

By [7, Chapter II], the algebra $\mathscr{A}_{G_1}$ is canonically identified with an algebra defined in terms of representations of ${}^L G_1$. Fix a finite unramified extension F'/F splitting G_1, and let σ be the q-Frobenius generator in $\mathrm{Gal}(F'/F)$. Since F' splits G_1, we may form ${}^L G_1$ using $\mathrm{Gal}(F'/F)$. We introduce the notation ${}^L G_1^{\mathrm{ur}}$ to emphasize this.

$${}^L G_1^{\mathrm{ur}} := \hat{G}_1 \rtimes \mathrm{Gal}(F'/F).$$

Following the notation of [71, §2], we define $\mathrm{ch}({}^L G_1^{\mathrm{ur}})$ to be the algebra consisting of restrictions of characters of finite dimensional representations of ${}^L G_1^{\mathrm{ur}}$ to the set of semi-simple $\hat{G}_1$-conjugacy classes in $\hat{G}_1 \rtimes \sigma$ (i.e. σ-conjugacy classes in $\hat{G}_1$). Then there is a canonical isomorphism $\mathscr{A}_{G_1} \cong \mathrm{ch}({}^L G_1^{\mathrm{ur}})$. Hence we may also view the Satake isomorphism as a canonical isomorphism

$$\mathcal{S}^{G_1} : \mathcal{H}^{\mathrm{ur}}(G_1(F)) \xrightarrow{\sim} \mathrm{ch}({}^L G_1^{\mathrm{ur}}). \tag{5.2.5}$$

Definition 5.2.3. In the following we will call (5.2.3) or (5.2.4) or (5.2.5) the *canonical Satake isomorphism.*

Next we recall the main result of [36, §2]:

Theorem 5.2.4 (Kottwitz)**.** *Let μ_1 be a cocharacter of G_1 defined over F. Assume μ_1 is* minuscule *in the sense that the representation* $\mathrm{Ad} \circ \mu_1$ *of* $\mathbb{G}_m$ *on* $\mathrm{Lie}\, G_{1,\bar{F}}$ *has no weights other than $-1, 0, 1$. For any hyperspecial subgroup K_1 of $G_1(F)$, let $f_{\mu_1,K_1} \in \mathcal{H}(G_1(F)//K_1)$ be the characteristic function of $K_1\mu_1(\varpi_F)K_1$ inside $G_1(F)$. Then the family $(f_{\mu_1,K_1})_{K_1}$ is compatible in the sense that it defines an element f_{μ_1} in $\mathcal{H}^{\mathrm{ur}}(G_1(F))$. Moreover, for any* *maximal split torus S inside G_1, we have*

$$\mathcal{S}_S^{G_1}(f_{\mu_1}) = q^{\langle \delta,\ \mu_{1,\mathrm{dom}} \rangle} \sum_{\mu_1' \in \Omega(F)\mu_{1,\mathrm{dom}}} \mu_1' \in \mathbb{C}[X_*(S)]^{\Omega(F)},$$

where T is the centralizer of S in G_1, δ is the half sum of the positive roots of T with respect to a fixed order, and $\mu_{1,\mathrm{dom}}$ is a cocharacter in $X_*(S)$ that is $G_1(F)$-conjugate to μ_1 and dominant with respect to the positive roots of T.

Next we discuss the compatibility between the Satake isomorphism and constant terms. Let S be a maximal split torus in G_1 and K_1 a hyperspecial subgroup determined by a hyperspecial vertex in the apartment of S. Let T be the centralizer of S. Let M be a Levi subgroup of a parabolic subgroup P of G. Assume $M \supset T$. Let N be the unipotent radical of P. Then $M(F) \cap K_1$ is a hyperspecial subgroup of $M(F)$. We define the constant term map

$$(\cdot)_M : \mathcal{H}(G_1(F)//K_1) \to \mathcal{H}(M(F)//M(F) \cap K_1) \tag{5.2.6}$$

$$f \mapsto f_M,\ f_M(m) = \delta_{P(F)}^{-1/2}(m) \int_{N(F)} f(nm)dn,\ m \in M(F),$$

where the Haar measure dn on $N(F)$ is normalized by the condition that $N(F) \cap K$ has volume 1.

Remark 5.2.5. The above definition is a special case of the general definition of the constant term map, recalled for instance in [18, §7.13], [71, §6.1]. For $M = T$, (5.2.6) is by definition the same as $\mathcal{S}_S^G$. In [71] (5.2.6) is called a *partial Satake transform.*

Lemma 5.2.6. *We have a commutative diagram:*

$$\begin{tikzcd} \mathcal{H}(G_1(F)//K_1) \arrow[r, "\mathcal{S}_S^{G_1}"] \arrow[d, "(\cdot)_M"] & \mathbb{C}[X_*(S)]^{\Omega(F)} \arrow[d] \\ \mathcal{H}(M(F)//M(F)\cap K_1) \arrow[r, "\mathcal{S}_S^M"] & \mathbb{C}[X_*(S)]^{\Omega_M(F)} \end{tikzcd}$$

where $\Omega_M(F)$ is the relative Weyl group of S in M, the horizontal arrows are the Satake transforms for G_1 and M respectively, and the right vertical arrow is the natural inclusion in view of the fact that $\Omega_M(F)$ is a subgroup of $\Omega(F)$.

Proof. This is well known and follows from the formulas (5.2.1) (5.2.6). See also [19, §12.3] or [71, §2, §6]. □

Corollary 5.2.7. *The constant term maps (5.2.6) lift to a canonical map that only depends on $M \subset G_1$:*

$$(\cdot)_M : \mathcal{H}^{\mathrm{ur}}(G_1(F)) \to \mathcal{H}^{\mathrm{ur}}(M(F)).$$

Remark 5.2.8. Under the canonical Satake isomorphisms, the map

$$(\cdot)_M : \mathcal{H}^{\mathrm{ur}}(G_1(F)) \to \mathcal{H}^{\mathrm{ur}}(M(F))$$

becomes the map $\mathscr{A}_{G_1} \to \mathscr{A}_M$ or the map $\mathrm{ch}({}^LG_1^{\mathrm{ur}}) \to \mathrm{ch}({}^LM^{\mathrm{ur}})$ induced by the embedding ${}^LM \hookrightarrow {}^LG_1$ which is canonical up to $\hat{G}$-conjugation.

5.3 Twisted transfer map

We recall the formalism of the twisted transfer map. Let G_1 be an unramified reductive group over $\mathbb{Q}_p$. Let $a \in \mathbb{Z}_{\geq 1}$ and let $F = \mathbb{Q}_{p^a}$ be the degree a unramified extension of $\mathbb{Q}_p$. Let σ be the p-Frobenius. Consider an elliptic endoscopic datum $(H, \mathcal{H}, s, \eta)$ of H. For simplicity, assume that $\mathcal{H} = {}^LH$, and also that $s \in Z(\hat{H})^{\Gamma_p}$. We also assume that the datum

(H, s, η) is unramified. Namely, H is unramified, and η is an L-embedding ${}^L H^{\mathrm{ur}} \to {}^L G^{\mathrm{ur}}$. Here we have denoted by ${}^L H^{\mathrm{ur}}$ and ${}^L G^{\mathrm{ur}}$ the L-groups formed by $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ or a large enough finite quotient of it.

Let $R := \mathrm{Res}_{F/\mathbb{Q}_p} G_1$. Then $\hat{R} = \prod_{i=1}^{a} \hat{G}_1$, and σ acts on $\hat{R}$ by the formula

$$\sigma(x_1, \cdots, x_a) \mapsto (\sigma x_2, \sigma x_3, \cdots, \sigma x_{a-1}, \sigma x_1).$$

Define

$$\tilde{\eta} : {}^L H^{\mathrm{ur}} \to {}^L R^{ur}$$

by

$$\hat{H} \ni x \mapsto (\eta(x), \cdots, \eta(x)) \rtimes 1$$

$$\sigma \mapsto (s^{-1}\eta(\sigma)\sigma^{-1}, \eta(\sigma)\sigma^{-1}, \cdots, \eta(\sigma)\sigma^{-1}) \rtimes \sigma. \tag{5.3.1}$$

Definition 5.3.1. The twisted transfer map $\mathcal{H}^{\mathrm{ur}}(G_1(F)) = \mathcal{H}^{\mathrm{ur}}(R(\mathbb{Q}_p)) \to \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p))$ is the map which under the canonical Satake isomorphisms corresponds to

$$(\tilde{\eta})^* : \mathrm{ch}({}^L R^{\mathrm{ur}}) \to \mathrm{ch}({}^L H^{\mathrm{ur}}).$$

Remark 5.3.2. $\tilde{\eta}$ is obtained by twisting the composition ${}^L H^{\mathrm{ur}} \xrightarrow{\eta} {}^L G_1^{\mathrm{ur}} \xrightarrow{\Delta} {}^L R^{\mathrm{ur}}$ by the 1-cocycle $\sigma \mapsto (s, 1, \cdots, 1)$ of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ in $\hat{R}$. Here Δ sends $x \in \hat{G}_1$ to $(x, \cdots, x) \in \hat{R}$ and sends σ to σ.

Remark 5.3.3. For the definition in more general situations, cf. [40, §7] or [32].

Remark 5.3.4. The formula (5.3.1) can be replaced by

$$\sigma \mapsto (t_1\eta(\sigma)\sigma^{-1}, t_2\eta(\sigma)\sigma^{-1}, \cdots, t_a\eta(\sigma)\sigma^{-1})$$

for any choice of $t_1, \cdots, t_a \in Z(\hat{H})^{\Gamma_p}$ such that $t_1 t_2 \cdots t_a = s^{-1}$. We have chosen $t_1 = s^{-1}, t_2 = t_3 = \cdots = t_a = 1$ for definiteness.

Next we would like to explicate the map $(\tilde{\eta})^*$ in view of the canonical isomorphisms $\mathrm{ch}({}^L R^{\mathrm{ur}}) \cong \mathscr{A}_R \cong \mathscr{A}_{G_{1,F}}$ and $\mathrm{ch}({}^L G_1^{\mathrm{ur}}) \cong \mathscr{A}_{G_1}$. Thus we fix a maximal $\mathbb{Q}_p$-split torus S_H in H and let T_H be its centralizer in H. Since G_1 is quasi-split, we know that T_H transfers to a maximal torus T in G_1. In particular, there is a Galois equivariant isomorphism

$$\iota : X_*(T_H) \xrightarrow{\sim} X_*(T) \tag{5.3.2}$$

canonical up to the action of $\Omega_H(\mathbb{Q}_p)$ on the left hand side and the action of $\Omega(\mathbb{Q}_p)$ on the right hand side. Here $\Omega_H(\mathbb{Q}_p)$ (resp. $\Omega(\mathbb{Q}_p)$) is the Galois fixed subgroup of the absolute Weyl group of T_H in H (resp. T in G_1), which is the same as the relative Weyl group of the maximal split subtorus of T_H (resp. T). Such an ι is called an admissible isomorphism from T_H to T. Moreover, whenever an admissible ι is chosen, there is a resulting embedding $\iota_* : \Omega_H(\mathbb{Q}_p) \hookrightarrow \Omega(\mathbb{Q}_p)$ such that ι is equivariant with respect to it.

Let S_H (resp. S) be the maximal split subtorus of T_H (resp. T). Then S_H (resp. S) is a maximal split torus in H (resp. G_1). Let S' be the maximal F-split subtorus of S. Then S' is a maximal F-split torus in $G_{1,F}$. We view both $X_*(S')$ and $X_*(S)$ naturally as subgroups of $X_*(T)$. Note that they are respectively the fixed point of σ^a and of σ acting on $X_*(T)$. The element $s \in Z(\hat{H})^{\Gamma_p}$ defines a canonical homomorphism $\langle \cdot, s \rangle : X_*(T_H) \to \mathbb{C}^\times$ that is $\Omega_H(\mathbb{Q}_p)$-equivariant. Fix a choice of ι, and consider the map

$$f_\iota : \mathbb{C}[X_*(S')]^{\Omega(F)} \to \mathbb{C}[X_*(T_H)]$$

$$\sum_{\chi \in X_*(S')} c_\chi [\chi] \mapsto \sum_{\chi \in X_*(S')} c_\chi \langle \iota^{-1}\chi, s \rangle [\iota^{-1}(\chi + \sigma\chi + \cdots + \sigma^{a-1}\chi)].$$

Lemma 5.3.5. *f_ι is independent of the choice of ι among admissible isomorphisms. More-*

over, f_ι has its image in $\mathbb{C}[X_*(S_H)]^{\Omega_H(\mathbb{Q}_p)}$.

Proof. Let ι' be another admissible isomorphism as in (5.3.2). Then there exist $\omega_0 \in \Omega(\mathbb{Q}_p)$ and $\omega_1 \in \Omega_H(\mathbb{Q}_p)$ such that $\iota' = \omega_0 \circ \iota \circ \omega_1$. Note that for $\omega_2 := \iota_*(\omega_1) \in \Omega(\mathbb{Q}_p)$, we have $\iota' = \omega_0 \circ \omega_2 \circ \iota$. Thus we may simplify notation and assume $\iota' = \omega_0 \circ \iota$ for some $\omega_0 \in \Omega(\mathbb{Q}_p)$. Take an arbitrary element $x = \sum_{\chi \in X_*(S')} c_\chi [\chi] \in \mathbb{C}[X_*(S')]^{\Omega(F)}$. Then we have $c_{\omega\chi} = c_\chi, \forall \chi \in X_*(S'), \forall \omega \in \Omega(F)$. For $\chi \in X_*(S')$, write $\mathrm{N}\,\chi := \chi + \sigma\chi + \cdots + \sigma^{a-1}\chi$. We compute

$$f_{\iota'}(x) = \sum_\chi c_\chi \langle \iota^{-1}\omega_0^{-1}\chi, s\rangle [\iota^{-1}\omega_0^{-1}\,\mathrm{N}\,\chi] = \sum_\chi c_\chi \langle \iota^{-1}\omega_0^{-1}\chi, s\rangle [\iota^{-1}\,\mathrm{N}(\omega_0^{-1}\chi)]$$

$$= \sum_\chi c_{\omega_0\chi} \langle \iota^{-1}\chi, s\rangle [\iota^{-1}\,\mathrm{N}\,\chi] \xlongequal{c_{\omega_0\chi}=c_\chi \text{ because } \omega_0\in\Omega(\mathbb{Q}_p)\subset\Omega(F)} \sum_\chi c_\chi \langle \iota^{-1}\chi, s\rangle [\iota^{-1}\,\mathrm{N}\,\chi] = f_\iota(x).$$

Now let $\omega_1 \in \Omega_H(\mathbb{Q}_p)$ be arbitrary and let $\omega_2 := \iota_*(\omega_1) \in \Omega(\mathbb{Q}_p)$. We compute

$$\omega_1(f_\iota(x)) = \sum_\chi c_\chi \langle \iota^{-1}\chi, s\rangle [\omega_1\iota^{-1}\,\mathrm{N}\,\chi] = \sum_\chi c_\chi \langle \iota^{-1}\chi, s\rangle [\iota^{-1}\omega_2\,\mathrm{N}\,\chi] =$$

$$= \sum_\chi c_\chi \langle \iota^{-1}\chi, s\rangle [\iota^{-1}\,\mathrm{N}(\omega_2\chi)] = \sum_\chi c_{\omega_2\chi} \langle \omega_1\iota^{-1}\chi, s\rangle [\iota^{-1}\,\mathrm{N}\,\chi],$$

which is equal to $f_\iota(x)$ because $c_{\omega_2\chi} = c_\chi$ and $\langle \omega_1\iota^{-1}\chi, s\rangle = \langle \iota^{-1}\chi, s\rangle$ by the $\Omega_H(\mathbb{Q}_p)$-equivariance of the map $\langle \cdot, s\rangle$. Finally, note that for each $\chi \in X_*(S') \subset X_*(T)$, we know that χ is fixed by σ^a, and so $\mathrm{N}\,\chi$ is fixed by σ, i.e. $\mathrm{N}\,\chi \in X_*(S)$. It follows that $\iota^{-1}\,\mathrm{N}\,\chi \in X_*(S_H)$, and so f_ι has its image in $\mathbb{C}[X_*(S_H)]^{\Omega_H(\mathbb{Q}_p)}$. □

Lemma 5.3.6. *Under the canonical isomorphisms*

$$\mathrm{ch}({}^L R^{\mathrm{ur}}) \cong \mathrm{ch}({}^L G_{1,F}^{\mathrm{ur}}) \cong \mathbb{C}[X_*(S')]^{\Omega(F)}$$

and

$$\mathrm{ch}({}^L H^{\mathrm{ur}}) \cong \mathbb{C}[X_*(S_H)]^{\Omega_H(\mathbb{Q}_p)},$$

the map

$$(\tilde{\eta})^* : \mathrm{ch}({}^L R^{\mathrm{ur}}) \to \mathrm{ch}({}^L H^{\mathrm{ur}})$$

corresponds to the map

$$f_\iota : \mathbb{C}[X_*(S')]^{\Omega(F)} \to \mathbb{C}[X_*(S_H)]^{\Omega_H(\mathbb{Q}_p)}.$$

Proof. This follows from the definition of $\tilde{\eta}$ and the various definitions in [7, §6]. □

5.4 Computation of twisted transfers

Recall the flag (1.1.1) inside V. Recall that we have fixed a splitting of it and denoted by W_i the quadratic space $V_i^\perp/V_i$, for $i = 1, 2$. Also recall the fixed vectors e_1, e_2, e_1', e_2' in §1.1.

Fix an odd prime p at which $G = \mathrm{SO}(V)$ is unramified. In the even case note that the discriminant δ of V has even p-adic valuation, by Proposition 3.2.5. Then all the standard Levi subgroups M_1, M_2, M_{12} are unramified too. Let M be one of them. We have a decomposition $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$, where $M^{\mathrm{GL}} = \mathrm{GL}_2$ or $\mathbb{G}_m$ or $\mathbb{G}_m^2$, and $M^{\mathrm{SO}} = \mathrm{SO}(W_1)$ or $\mathrm{SO}(W_2)$.

Consider an elliptic G-endoscopic datum for M of the form $\mathfrak{e}_{A,d^+,d^-}(M,G)$ (resp. of the form $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(M,G)$) in the odd case (resp. the even case.) To ease notation we always write $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}$, omitting (M,G) from the notation and understanding that in the odd case $\delta^+ = \delta^- = 1$. The associated endoscopic datum for G is $\mathfrak{e}_{d^++2|A|,\delta^+,d^-+2|A^c|,\delta^-}$ and the underlying endoscopic datum for M is $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M)$, which we abbreviate as $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$. Note that $\mathfrak{e}_{d^++2|A|,\delta^+,d^-+2|A^c|,\delta^-}$ is unramified if and only if $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}$ is unramified if and only if both δ^+ and δ^- have even p-adic valuation. (Note that the in the odd case both the data $\mathfrak{e}_{d^++2|A|,d^-+2|A^c|}$ and $\mathfrak{e}_{d^+,d^-}$ are automatically unramified.)

Write

$$\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-} = (M', s_M, \eta_M)$$

$$\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-} = (M', s'_M, \eta_M)$$

$$\mathfrak{e}_{d^++2|A|,\delta^+,d^-+2|A^c|,\delta^-} = (H, s, \eta).$$

Recall that s_M is of the form

$$s_M = (s^{\mathrm{GL}}, s^{\mathrm{SO}}) \in \widehat{M^{\mathrm{GL}}} \times \widehat{M^{\mathrm{SO}}},$$

and s'_M is of the form

$$s_M = (1, s^{\mathrm{SO}}) \in \widehat{M^{\mathrm{GL}}} \times \widehat{M^{\mathrm{SO}}}.$$

Recall that the component $s^{\mathrm{GL}} \in \widehat{M^{\mathrm{GL}}}$ is determined by the parameter A, in the way which can be roughly summarized as saying that A records the positions of 1's in s^{GL} and in the rest positions there are -1's.

Since M' is naturally a Levi subgroup of H, we have the canonical constant term map (cf. §5.2):

$$(\cdot)_{M'} : \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p)) \to \mathcal{H}^{\mathrm{ur}}(M'(\mathbb{Q}_p)).$$

Fix $a \in \mathbb{Z}_{\geq 1}$. We have the twisted transfer map induced by the unramified endoscopic datum (H, η, s) for G (cf. §5.3):

$$b : \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{p^a})) \to \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p)).$$

Let μ be the identity cocharacter for the first $\mathbb{G}_m$ (i.e. $\mathrm{GL}(V_2/V_1)$) in $M_{12} = \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{SO}(W_2)$. This is a minuscule cocharacter of G defined over $\mathbb{Q}_p$. In fact, μ is a Hodge cocharacter of the Shimura datum $\mathbf{O}(V)$ (§1.2).

Definition 5.4.1. Define $f_{-\mu} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{p^a}))$ as in Theorem 5.2.4. Define $f^H := b(f_{-\mu}) \in \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p))$.

Lemma 5.4.2. *Extend $M_{12}^{\mathrm{GL}} = \mathbb{G}_m^2$ to a maximal torus T in G (defined over $\mathbb{Q}_p$), of the form $T = \mathbb{G}_m^2 \times T_{W_2}$ where T_{W_2} is a maximal torus in $M_{12}^{\mathrm{SO}} = \mathrm{SO}(W_2)$. Let Ω be the absolute Weyl group of T in G and let $\Omega(\mathbb{Q}_p)$ be the relative Weyl group of the maximal split subtorus of T (for the ground field $\mathbb{Q}_p$). Let $\mathcal{O}$ (resp. $\mathcal{O}_0$) be the Ω-orbit (resp. $\Omega(\mathbb{Q}_p)$-orbit) of the cocharacter $-\mu$ of T. Then*

$$\mathcal{O} = (\mathcal{O}_0 \cap X_*(\mathbb{G}_m^2)) \sqcup (\mathcal{O} \cap X_*(T_{W_2}))$$

and

$$\mathcal{O}_0 \cap X_*(\mathbb{G}_m^2) = \{\epsilon_1^\vee, -\epsilon_1^\vee, \epsilon_2^\vee, -\epsilon_2^\vee\}.$$

Proof. Let $m = \lfloor d/2 \rfloor$. As usual we let $\epsilon_1^\vee, \epsilon_2^\vee$ be the standard basis of $X_*(\mathbb{G}_m^2)$, and find a basis $\epsilon_3^\vee, \cdots, \epsilon_m^\vee$ for $X_*(T_{W_2})$, such that the coroots in

$$X_*(T) \cong X_*(\mathbb{G}_m^2) \oplus X_*(T_{W_2})$$

are given by

$$\pm\epsilon_i^\vee \pm \epsilon_j^\vee, i \neq j, \quad \pm 2\epsilon_i^\vee$$

in the odd case, and

$$\pm\epsilon_i^\vee \pm \epsilon_j^\vee, i \neq j$$

in the even case. Using the basis $\epsilon_1^\vee, \cdots, \epsilon_m^\vee$ to identify $X_*(T)$ with $\mathbb{Z}^m$, the action of Ω becomes the natural action of $\{\pm 1\}^m \rtimes \mathfrak{S}_m$ (resp. $(\{\pm 1\}^m)' \rtimes \mathfrak{S}_m$) on $\mathbb{Z}^m$ in the odd case (resp. even case). Here $(\{\pm 1\}^m)'$ denotes the subgroup of $\{\pm 1\}^m$ consisting of elements with an even number of -1's in the coordinates. We have $-\mu = -\epsilon_1^\vee$. Hence it is clear that

$$\mathcal{O} = (\mathcal{O} \cap X_*(\mathbb{G}_m^2)) \sqcup (\mathcal{O} \cap X_*(T_{W_2}))$$

and that

$$\mathcal{O} \cap X_*(\mathbb{G}_m^2) = \{\epsilon_1^\vee, -\epsilon_1^\vee, \epsilon_2^\vee, -\epsilon_2^\vee\}.$$

It suffices to show that the above set can be obtained by translating $-\epsilon_1^\vee$ under Ω_0. But clearly the above set is equal to the orbit of $-\epsilon_1^\vee$ under the group generated by the reflections $s_{\epsilon_1^\vee - \epsilon_2^\vee}$ and $s_{\epsilon_1^\vee + \epsilon_2^\vee}$. Note that the two coroots $\epsilon_1^\vee + \epsilon_2^\vee$ and $\epsilon_1^\vee - \epsilon_2^\vee$ are both defined over $\mathbb{Q}_p$, so their associated reflections are in $\Omega(\mathbb{Q}_p)$. $\square$

Since M^{GL} is $\mathbb{G}_m$ or GL_2 or $\mathbb{G}_m^2$, its Satake isomorphism (over $\mathbb{Q}_p$) is from $\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p))$ to $\mathbb{C}[X_1]$ or $\mathbb{C}[X_1, X_2]^{\mathfrak{S}_2}$ or $\mathbb{C}[X_1, X_2]$ respectively. To ease notation we will identify an element of the latter algebras $\mathbb{C}[X_1]$ etc. with an element of $\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p))$. Recall that $M' = M^{\mathrm{GL}} \times (M')^{\mathrm{SO}}$, and therefore we have

$$\mathcal{H}^{\mathrm{ur}}(M'(\mathbb{Q}_p)) = \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p)) \otimes_{\mathbb{C}} \mathcal{H}^{\mathrm{ur}}((M')^{\mathrm{SO}}(\mathbb{Q}_p)).$$

Proposition 5.4.3. *The function $p^{a(2-d)/2}(f^H)_{M'} \in \mathcal{H}^{\mathrm{ur}}(M'(\mathbb{Q}_p))$ is equal to a sum*

$$k(A) \otimes 1 + 1 \otimes h,$$

with $k(A) \in \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p))$ and $h \in \mathcal{H}^{\mathrm{ur}}((M')^{\mathrm{SO}}(\mathbb{Q}_p))$. The element h depends only on the parameters $d^+, \delta^+, d^-, \delta^-$, and not on the parameter A. The element $k(A)$ is given as follows:

1. *When $M = M_{12}$, so that $M^{\mathrm{GL}} = \mathbb{G}_m^2$ and $A \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$,*

$$k(A) = \epsilon_1(A)(X_1^a + X_1^{-a}) + \epsilon_2(A)(X_2^a + X_2^{-a}),$$

where $\epsilon_i(A) \in \pm 1$ and $\epsilon_i(A) = 1$ if and only if $i \in A$, for $i = 1, 2$.

2. *When $M = M_1$, so that $M^{\mathrm{GL}} = \mathrm{GL}_2$ and $A \in \{\emptyset, \{1,2\}\}$,*

$$k(A) = \begin{cases} X_1^a + X_1^{-a} + X_2^a + X_2^{-a}, \ A = \{1,2\} \\ -X_1^a - X_1^{-a} - X_2^a - X_2^{-a}, \ A = \emptyset \end{cases}.$$

3. *When $M = M_2$, so that $M^{\mathrm{GL}} = \mathbb{G}_m$ and $A \in \{\emptyset, \{1\}\}$, we have*

$$k(A) = \begin{cases} X_1^a + X_1^{-a}, \ A = \{1\} \\ -X_1^a - X_1^{-a}, \ A = \emptyset \end{cases}.$$

Proof. This follows directly from Theorem 5.2.4, Lemma 5.2.6, Lemma 5.3.6, Lemma 5.4.2.

□

6 Stabilization

6.1 Standard definitions and facts in Langlands-Shelstad transfer

Let $F = \mathbb{Q}_v$ where v is a finite place of $\mathbb{Q}$. Let G_1 be a reductive group over F. Fix an inner twisting $\psi : G_1 \to G_1^*$ to the quasi-split inner form and fix an L-group datum for G_1, as in [53]. Let $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum for G_1. Assume that $\mathcal{H} = {}^L H$. This last assumption is satisfied for the applications in this work, and simplifies many aspects of the theory. Most significantly, we do not need to take z-extensions of G_1 and H in the statement of transfer of orbital integrals.

We first recall the definition of κ-orbital integrals in the fashion of [43, §2.7].

Let $\gamma \in G_1(F)$ be semi-simple. Denote $I_\gamma = (G_1)^0_\gamma$. Define

$$\mathfrak{D}(I_\gamma, G_1; F) := \ker(\mathbf{H}^1(F, I_\gamma) \to \mathbf{H}^1(F, G_1)).$$

Recall (cf. [43] for instance) that there is a finite-to-one surjective map from $\mathfrak{D}(I_\gamma, G_1; F)$ to the set of conjugacy classes in the stable conjugacy class of γ. When $I_\gamma = (G_1)_\gamma$, this last map is a bijection, but we will not assume that. We have a short exact sequence of pointed sets:

$$1 \to I_\gamma(F)\backslash G_1(F) \to \mathbf{H}^0(F, I_\gamma\backslash G_1) \to \mathfrak{D}(I_\gamma, G_1; F) \to 1. \tag{6.1.1}$$

Also we have a natural map (cf. loc. cit.)

$$\mathbf{H}^0(F, I_\gamma\backslash G_1) \to \mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G_1),$$

where $\mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G_1)$ is a locally compact topological abelian group. Denote by $\mathfrak{K}(I_\gamma, G; F)$ the Pontryagin dual of $\mathbf{H}^0_{\mathrm{ab}}(F, I_\gamma\backslash G_1)$ [17]

We fix a measure dx on $\mathbf{H}^0(F, I_\gamma\backslash G_1)$ by choosing Haar measures on $I_\gamma(F)$ and $G_1(F)$ and choosing the counting measure on $\mathfrak{D}(I_\gamma, G_1; F)$ and require compatibility of the sequence (6.1.1). For more details see [43, §2.7].

For any $f \in C_c^\infty(G_1(F))$, any $\kappa \in \mathfrak{K}(I_\gamma, G_1; F)$, define the κ-orbital integral

$$O^\kappa_\gamma(f) := \int_{x \in \mathbf{H}^0(\mathbb{A}_f, I_\gamma\backslash G_1)} e((G_1)^0_{x^{-1}\gamma x})\kappa(x) f(x^{-1}\gamma x) dx.$$

[17] We have assumed that F is non-archimedean. The group $\mathfrak{K}(I_\gamma, G; F)$ is isomorphic to Kottwitz's $\mathfrak{K}(I_\gamma/F)$ in [38, §4.6]. In the archimedean case we may still define $\mathfrak{K}(I_\gamma, G; F)$ in the same way, but it will differ from the definition loc. cit.

Define

$$SO_\gamma(f) := O^1_\gamma(f).$$

More concretely, for any $[x] \in \mathfrak{D}(I_\gamma, G_1; F)$, fix an element $x \in G_1(\bar{F})$ mapping to $[x]$ under the composition

$$G_1(\bar{F}) \to \mathbf{H}^0(F, I_\gamma \backslash G_1) \to \mathfrak{D}(I_\gamma, G_1; F).$$

Then $\gamma_x := x^{-1}\gamma x$ is in $G_1(F)$ and $\mathrm{Int}(x)$ induces an inner twisting $I_\gamma \to I_{\gamma_x}$. In particular, a Haar measure on $I_{\gamma_x}(F)$ is fixed from that on $I_\gamma(F)$, and we define

$$O_{\gamma_x}(f) := \int_{I_{\gamma_x}(F)\backslash G_1(F)} f(y^{-1}\gamma_x y)dy,$$

where dy is determined by the fixed Haar measure on $G_1(F)$ and the Haar measure on $I_{\gamma_x}(F)$ mentioned above. Then we have

$$O^\kappa_\gamma(f) = \sum_{[x]\in\mathfrak{D}(I_\gamma, G_1; F)} e(I_{\gamma_x})\kappa(x) O_{\gamma_x}(f).$$

Recall that the notion of when a semi-simple element $\gamma_H \in H(F)$ (not necessarily G_1-regular) is an image of a semi-simple element $\gamma \in G_1(F)$ is defined in [48, §1.2]. For (G_1, H)-regular elements, Langlands-Shelstad [53] [48] define transfer factors. Thus after choosing a common normalization we have numbers

$$\Delta(\gamma_H, \gamma) \in \mathbb{C},$$

where γ_H is a semi-simple (G_1, H)-regular element in $H(F)$, up to stable conjugacy, and γ is a semi-simple element in $G_1(F)$, up to conjugacy, and $\Delta(\gamma_H, \gamma) = 0$ unless γ_H is an image of γ.[18]

The Langlands-Shelstad Transfer Conjecture and the Fundamental Lemma are now proved theorems thanks to the work of Ngô [61], Waldspurger [79] [80], Cluckers-Loeser [12], and Hales[21]. Combined with the extension to (G_1, H)-regular elements in [48, §2.4], we state them in the following form:

Proposition 6.1.1. *The following statements are true.*

1. *Fix a normalization of the transfer factors. Fix Haar measures on $G_1(F)$ and $H(F)$. For any $f \in C_c^\infty(G_1(F))$, there exists $f^H \in C_c^\infty(H(F))$ (depending on the Haar measures on $G_1(F)$ and $H(F)$), called the* Langlands-Shelstad transfer *of f, with the following properties: For any semi-simple (G_1, H)-regular element γ_H in $H(F)$, we have*

$$(6.1.2) \qquad SO_{\gamma_H}(f^H) = \begin{cases} 0, & \gamma_H \text{ is not an image from } G_1 \\ \Delta(\gamma_H, \gamma) O_\gamma^s(f), & \gamma_H \text{ is an image of } \gamma \in G_1(F)_{\mathrm{ss}}. \end{cases}$$

   where in the second situation,

   - *s defines an element in $\mathfrak{K}(I_\gamma, G_1; F)$, also denoted by s.*
   - *Both $SO_{\gamma_H}(f^H)$ and $O_\gamma^s(f)$ are computed using the fixed Haar measures on $G_1(F)$ and $H(F)$ and* compatible Haar measures *on $(G_1)_\gamma^0(F)$ and $H_{\gamma_H}^0(F)$, where compatible means with respect to the natural inner twisting between $(G_1)_\gamma^0$ and $H_{\gamma_H}^0(F)$.*

[18] Strictly speaking, Langlands-Shelstad only define transfer factors in the generality of (G_1, H)-regular elements under the assumption that G_1^{der} is simply connected ([48, §2.4]). However under our assumption that $\mathcal{H} = {}^L H$, we easily get the definition of $\Delta(\gamma_H, \gamma)$ by considering suitable z-extensions $\tilde{H}$ and $\tilde{G}_1$ of H and G_1 by a common kernel Z and defining $\Delta(\gamma_H, \gamma)$ to be $\Delta(\tilde{\gamma}_H, \tilde{\gamma})$ where $\tilde{\gamma}_H$ (resp. $\tilde{\gamma}$) is a preimage of γ_H (resp. γ) in $\tilde{H}(F)$ (resp. $\tilde{G}_1(F)$). To show that this definition is independent of the choices of the preimages, it suffices to check that the transfer factor $\Delta(\tilde{\gamma}_H, \tilde{\gamma})$ depends only on $\tilde{\gamma}_H$ modulo $Z(F)$ and $\tilde{\gamma}$ modulo $Z(F)$. For this it suffices to treat the case where $\tilde{\gamma}_H$ is strongly $\tilde{G}_1$-regular. Then the desired statement is proved in [48, p. 55] (with $\lambda = 1$). See also Remark 6.1.3 below. It can also be shown that the definition of $\Delta(\gamma_H, \gamma)$ is independent of the choices $\tilde{H}$ and $\tilde{G}_1$.

2. *Suppose G_1 and $(H, {}^L H, s, \eta)$ are unramified. Normalize the Haar measures on $G_1(F)$ and $H(F)$ such that all the hyperspecial subgroups have volume 1. Let K_1 (resp. K_H) be an arbitrary hyperspecial subgroup of $G_1(F)$ (resp. $H(F)$). Then 1_{K_H} is a Langlands-Shelstad transfer of 1_{K_1} as in part (1), for the canonical unramified normalization of transfer factors defined in [20].*

3. *Suppose G_1 is defined over $\mathbb{Q}$ and let $(H, {}^L H, s, \eta)$ be a global endoscopic datum. Suppose for almost all places at which G_1 and $(H, {}^L H, s, \eta)$ are unramified the transfer factor is normalized under the canonical unramified normalization. Let S be a finite set of places including ∞ and let $\mathbb{A}^S$ denote the adeles aways from S. For any $f \in C_c^\infty(G_1(\mathbb{A}^S))$, there exists $f^H \in C_c^\infty(H(\mathbb{A}^S))$ such that the $\mathbb{A}^S$-analogue of (6.1.2) holds. Here the meaning of an adelic (G_1, H)-regular element is as in [40, §7, pp. 178-179], and all the orbital integrals are computed with respect to adelic Haar measures.*

Remark 6.1.2. In Proposition 6.1.1 part (3) is a consequence of parts (1) and (2)

Remark 6.1.3. We have stated the Langlands-Shelstad Transfer Conjecture in a seemingly stronger form than the original form in two ways. Firstly, the original conjecture allows for a suitable z-extension of H when G_1^{der} is not simply connected, in order to take care of the failure of $\mathcal{H} = {}^L H$. Let us recall the setting. One considers z-extensions

$$1 \to Z \to \tilde{H} \to H \to 1$$

$$1 \to Z \to \tilde{G}_1 \to G_1 \to 1,$$

sharing the same central kernel Z, such that Z is a product of induced tori, and $\tilde{H}$ and $\tilde{G}_1$ have simply connected derived subgroups. One then asks to transfer functions in $C_c^\infty(G_1(F))$ to functions in $C_c^\infty(\tilde{H}, \lambda)$, where λ is a certain character on $Z(F)$. In our case, since we have assumed $\mathcal{H} = {}^L H$, it turns out that one may easily arrange $\lambda = 1$. Hence there is a canonical bijection $C_c^\infty(\tilde{H},1) \xrightarrow{\sim} C_c^\infty(H)$, such that a Langlands-Shelstad transfer of f in $C_c^\infty(\tilde{H},1)$ corresponds to a Langlands-Shelstad transfer of f in $C_c^\infty(\tilde{H},1)$ in the sense of Proposition 6.1.1.

Secondly, the original conjecture is only about G_1-regular elements. Namely, (6.1.2) is only required to hold for all G_1-regular elements. Let us call f^H satisfying (6.1.2) only for G_1-regular elements a *weaker transfer* of f. In [48, §2.4], Langlands-Shelstad proves that a weaker transfer in fact satisfies (6.1.2) for all (G_1, H)-regular elements, but under the assumption that G_1^{der} is simply connected. We do not assume G_1^{der} is simply connected, but we assume $\mathcal{H} = {}^L H$, and it is quite easy to reduce to the result of Langlands-Shelstad. In fact, keeping the notation in the previous paragraph, for any $f \in C_c^\infty(G_1(F))$ we define $\tilde{f} \in C^\infty(\tilde{G}_1(F))$ to be the pull back of f. Let $f^{\tilde{H}} \in C_c^\infty(\tilde{H}(F),1)$ be a weak transfer of f. Again define $f^H \in C_c^\infty(H(F))$ to be the function canonically induced by $f^{\tilde{H}}$. One easily checks that $f^{\tilde{H}}$ is a weak transfer of $\tilde{f}$, in the sense that there is a natural way of enhancing $\tilde{H}$ to an endoscopic datum of $\tilde{G}_1$. By [48, §2.4], $f^{\tilde{H}}$ and $\tilde{f}$ satisfy (6.1.2) for all $(\tilde{G}_1, \tilde{H})$-regular elements. From there it is easy to deduce that f^H and f satisfy (6.1.2) for all (G_1, H)-regular elements. Among the key points is the observation that we have surjections $\tilde{H}(F) \to H(F)$ and $\tilde{G}_1(F) \to G_1(F)$, and natural bijections

$$\mathfrak{D}(I_{\tilde{\gamma}}, \tilde{G}_1; F) \xrightarrow{\sim} \mathfrak{D}(I_\gamma, G_1; F)$$

$$\mathfrak{D}(I_{\gamma_{\tilde{H}}}, \tilde{H}; F) \xrightarrow{\sim} \mathfrak{D}(I_{\gamma_H}, H; F),$$

when we have semi-simple elements $\tilde{\gamma} \in \tilde{G}_1(F)$ mapping to $\gamma \in G_1(F)$ and $\gamma_{\tilde{H}} \in \tilde{H}(F)$ mapping to $\gamma_H \in H(F)$.

Similar remarks apply to part (2) of Proposition 6.1.1.

6.2 Calculation of τ and k

Let G_1 be the special orthogonal group of an arbitrary quadratic space over $\mathbb{Q}$. Let m_1 be the $\bar{\mathbb{Q}}$-rank of G_1.

Proposition 6.2.1. $\tau(G_1) = 2$.

Proof. By [37] and [39], $\tau(G_1) = \left|\pi_0(Z(\hat{G}_1)^\Gamma)\right| / \left|\ker^1(\mathbb{Q}, Z(\hat{G}_1))\right|$. Now $\hat{G}_1$ is a symplectic group or an even orthogonal group, so $Z(\hat{G}_1) \cong \mu_2$. Thus any group action on $Z(\hat{G}_1)$ must be trivial. In particular, $\ker^1(\mathbb{Q}, Z(\hat{G}_1)) = 0$ by Chebotarev density. Also $\pi_0(Z(\hat{G}_1)^\Gamma) = Z(\hat{G}_1)$ has cardinality 2. $\square$

Definition 6.2.2. Suppose $G_{1,\mathbb{R}}$ has elliptic maximal tori. Let T_e be one such. Define

$$k'(G_1) := \left|\mathbf{H}^1(\mathbb{R}, T_e)\right|,$$

$$k(G_1) := \left|\mathrm{im}(\mathbf{H}^1(\mathbb{R}, T_e \cap G_1^{\mathrm{SC}}) \to \mathbf{H}^1(\mathbb{R}, T_e))\right|.$$

Proposition 6.2.3. *Suppose $G_{1,\mathbb{R}}$ has elliptic maximal tori.* $k'(G_1) = 2^{m_1}$, $k(G_1) = 2^{m_1-1}$.

Proof. By Tate-Nakayama duality,

$$k'(G_1) = \left|\pi_0(\hat{T_e}^{\Gamma_\infty})\right| = \left|\widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(T_e))\right|.$$

As argued in the proofs of [59, Lemma 5.4.2] Lemma 5.4.2 and [60, Lemme 5.2.2], we have[19]

$$k(G_1) = \left|\pi_0(\hat{T_e}^{\Gamma_\infty})\right| / \left|\pi_0(Z(\hat{G}_1)^{\Gamma_\infty})\right|.$$

We have seen in the proof of the previous Proposition that $\left|\pi_0(Z(\hat{G}_1)^{\Gamma_\infty})\right| = \left|Z(\hat{G}_1)\right| = 2$. We know that $T_e \cong \mathrm{U}(1)^{m_1}$, and $X_*(\mathrm{U}(1)) \cong \mathbb{Z}$ on which the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$

[19] In loc. cit. $k(\cdot)$ is defined using G^{der} instead of G^{SC}, but $G^{\mathrm{der}} = G^{\mathrm{SC}}$ in that case, and this formula for $k(\cdot)$ generalizes to our definition of $k(\cdot)$ here.

acts as multiplication by -1. Hence

$$\left|\widehat{\mathbf{H}}^{-1}(\Gamma_\infty, X_*(T_e))\right| = X_*(T_e)^{\mathrm{N}} / \operatorname{Aug}_{\Gamma_\infty}(X_*(T_e)) = X_*(T_e)/2X_*(T_e) \cong (\mathbb{Z}/2\mathbb{Z})^{m_1}.$$

□

Proposition 6.2.4. *For any $j \geq 1$, $\tau(\mathrm{GL}_j) = 1$. For $j = 1, 2$, $k(\mathrm{GL}_j) = 1$. ($\mathrm{GL}_{j,\mathbb{R}}$ has elliptic maximal tori precisely when $j = 1, 2$.)*

Proof. For $j \geq 1$, $Z(\widehat{\mathrm{GL}}_j) = \mathbb{C}^\times$, on which Γ acts trivially. Hence

$$\pi_0(Z(\widehat{\mathrm{GL}}_j)^\Gamma) = \pi_0(\mathbb{C}^\times) = 1,$$

and

$$\ker^1(\mathbb{Q}, Z(\widehat{\mathrm{GL}}_j)) = 1$$

by Chebotarev density theorem. Thus $\tau(\mathrm{GL}_j) = 1$. Also

$$\pi_0(Z(\widehat{\mathrm{GL}}_j)^{\Gamma_\infty}) = 1.$$

For $j = 1$, $\mathrm{GL}_j^{\mathrm{SC}} = 1$, so $k(\mathrm{GL}_j) = 1$. (Alternatively, $T_e \cong \mathbb{G}_m$, so by Hilbert 90, $\mathbf{H}^1(\mathbb{R}, T_e) = 1$.) For $j = 2$, $T_e \cong \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, and $\mathbf{H}^1(\mathbb{R}, T_e) = 1$ by Shapiro's Lemma. □

Corollary 6.2.5. *Let M be a Levi subgroup of G defined over $\mathbb{Q}$. Let M' be the first entry of a G-endoscopic datum for M. Let H' be the first entry of the induced endoscopic datum for G. Then we have*

$$\frac{\tau(G)}{\tau(H)}\frac{\tau(M')}{\tau(M)} = \frac{k(H)}{k(G)}\frac{k(M)}{k(M')}.$$

Proof. First suppose $M' \cong M^*$. Then we need to prove that

$$\frac{\tau(G)}{\tau(H)} = \frac{k(H)}{k(G)}.$$

If $H \cong G^*$, then both sides are equal to 1. Otherwise, both sides are equal to 1/2.

Suppose $M' \ncong M^*$. Then $H \ncong G^*$. Hence $\tau(G) = 2, \tau(H) = 4, \tau(M') = 4, \tau(M) = 2$, and $k(H)/k(G) = k(M')/k(M) = 1/2$. The claim follows. $\square$

6.3 The simplified geometric side of the stable trace formula

We recall the definition, due to Kottwitz in his unpublished notes, of the simplified geometric side of the stable trace formula, applicable to the situation where the test function at infinity is stable cuspidal. We follow [59, §5.4]. In this work we do not prove the relation between this definition and Arthur's stabilization.

Definition 6.3.1. Let M be a cuspidal reductive group over $\mathbb{R}$. Fix a Haar measure on $M(\mathbb{R})$. Let $\bar{M}$ be the compact-mod-center inner form of M. Then there is an induced Haar measure on $\bar{M}(\mathbb{R})$. Define

$$\bar{v}(M) := e(\bar{M}) \operatorname{vol}(\bar{M}(\mathbb{R})/A_M(\mathbb{R})^0).$$

Definition 6.3.2. Let G_1 be a reductive group over $\mathbb{R}$. Let $\nu : A_{G_1}(\mathbb{R})^0 \to \mathbb{C}^\times$ be a quasi-character. Let M be a cuspidal Levi subgroup of G_1 , and $f \in C_c^\infty(G_1(\mathbb{R}), \nu^{-1})$ be stable cuspidal ([1, §4], [59, 5.4]). For $\gamma \in M(\mathbb{R})$ semi-simple elliptic, define

$$S\Phi_M^{G_1}(\gamma, f) := (-1)^{\dim A_M} k(M) k(G_1)^{-1} \bar{v}(M_\gamma^0)^{-1} \sum_{\Pi} \Phi_M(\gamma^{-1}, \Theta_\Pi) \operatorname{Tr}(f, \Pi),$$

where Π runs through the discrete series L-packets belonging to ν. This definition depends on the choice of a Haar measure on $M_\gamma^0(\mathbb{R})$ (to define $\bar{v}(M_\gamma^0)$) and a Haar measure on $G_1(\mathbb{R})$

(to define $\mathrm{Tr}(f,\Pi)$). For a non-cuspidal Levi subgroup M of G_1, define $S\Phi_M^{G_1} \equiv 0$.

Definition 6.3.3. Let G_1 be a reductive group over $\mathbb{Q}$. To avoid the issue discussed in Remark 4.1.2, assume that all the Levi subgroups M of G_1 defined over $\mathbb{Q}$ (including G_1 itself) satisfy the condition that $A_{M_{\mathbb{R}}} = (A_M)_{\mathbb{R}}$. Assume that G_1 is cuspidal. For $f = f^\infty f_\infty \in C_c^\infty(G_1(\mathbb{A}))$ with $f_\infty \in C_c^\infty(G_1(\mathbb{R}), \nu^{-1})$ stable cuspidal, and $M \subset G_1$ a $\mathbb{Q}$-Levi subgroup, define

$$ST_M^{G_1}(f) := \tau(M) \sum_\gamma \bar{\iota}^M(\gamma)^{-1} SO_\gamma(f_M^\infty) S\Phi_M^{G_1}(\gamma, f_\infty),$$

where γ runs through a set of representatives of the stable conjugacy classes of the $\mathbb{R}$-elliptic semi-simple elements of $M(\mathbb{Q})$, and

$$\bar{\iota}^M(\gamma) := \left|(M_\gamma/M_\gamma^0)(\mathbb{Q})\right|.$$

Here each $SO_\gamma(f_M^\infty)$ is defined using fixed Haar measures on $M_\gamma^0(\mathbb{A}_f)$ and on $M(\mathbb{A}_f)$, and $S\Phi_M^{G_1}(\gamma, f_\infty)$ is defined using fixed Haar measures on $M_\gamma^0(\mathbb{R})$ and on $G_1(\mathbb{R})$. We assume that the product measure on $M_\gamma^0(\mathbb{A})$ is the Tamagawa measure. Also, the constant term f_M^∞ is defined using the fixed Haar measure on $M(\mathbb{A}_f)$ mentioned above, and a fixed Haar measure on $G_1(\mathbb{A}_f)$. We assume that the product measure on $G_1(\mathbb{A})$ is the Tamagawa measure. Then $ST_M^{G_1}(f)$ is independent of the choices of Haar measures.

Definition 6.3.4. Let G_1 be as in Definition 6.3.3. Define

$$ST^{G_1}(f) := \sum_M (n_M^{G_1})^{-1} ST_M^{G_1}(f),$$

where M runs through the Levi subgroups of G_1 defined over $\mathbb{Q}$, up to $G_1(\mathbb{Q})$-conjugacy, and as before $n_M^{G_1} := \left|[\mathrm{Nor}_{G_1}(M)/M](\mathbb{Q})\right|$.

6.4 Test functions on endoscopic groups

As in Theorem 1.7.2, we fix an integer $a \geq 1$, an irreducible algebraic representation $\mathbb{V}$ of $G_{\mathbb{C}}$, an odd prime p, and a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. Also fix a test function $f^{p,\infty} \in C_c^\infty(G(\mathbb{A}_f^p))$, and fix a Haar measure dg^p on $G(\mathbb{A}_f^p)$.

Let $(H, {}^L H, s, \eta)$ be an elliptic endoscopic datum for G, presented in the explicit form as in §5.1. In the following we will avoid using the representative of the endoscopic data with $s = -1$. Namely, we will only use the trivial representative for the trivial endoscopic datum $H = G^*$. Later when we want to recall this assumption we simply say $s \neq -1$.

We follow Kottwitz [40] to define a test function $f^H \in C_c^\infty(H(\mathbb{A}))$. By definition f^H is of the form $f^H = f_\infty^H f_p^H f^{H,p,\infty}$, with $f_\infty^H \in C_c^\infty(H(\mathbb{R}))$ stable cuspidal. (As H is semi-simple we do not need central characters.) By definition, $f^H = 0$ unless H is cuspidal (at infinity) and unramified at p. Note that by our explicit presentation of $(H, {}^L H, s, \eta)$ in §5.1, the algebraic group H is unramified at p if and only if the endoscopic datum $(H, {}^L H, s, \eta)$ is unramified at p. In the following we assume that H is cuspidal and unramified at p.

We fix a Haar measure on $H(\mathbb{A}_f^p)$ arbitrarily, and fix the Haar measure on $H(\mathbb{Q}_p)$ such that hyperspecial subgroups have volume 1. Then there is a unique Haar measure on $H(\mathbb{R})$ such that the product measure on $H(\mathbb{A})$ is the Tamagawa measure.

We first recall the definition of f_∞^H. The definition of f_∞^H depends on the choice of $(j, B_{G,H})$. Here $j : T_H \to T_G$ is an admissible isomorphism between two fixed elliptic maximal tori T_H and T_G of $H_{\mathbb{R}}$ and $G_{\mathbb{R}}$ respectively, i.e. j is the composition of an admissible embedding $T_H \to G^*$ (which is defined over $\mathbb{R}$) and the fixed inner twisting $G^* \to G$ and a certain inner automorphism of $G_{\mathbb{C}}$, such that $j : T_H \to T_G$ is defined over $\mathbb{R}$. The choices of such j form a torsor under $\Omega = \Omega(G_{\mathbb{C}}, T_{G,\mathbb{C}})$[20]. Moreover, $B_{G,H}$ is a Borel of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$.

[20]Note that any algebraic group automorphism of $T_{G,\mathbb{C}}$ is necessarily defined over $\mathbb{R}$ because T_G is a product of U(1).

6.4.1 Fixing the choice of $(T_H, T_G, j, B_{G,H})$ once and for all

Choose $(T_G, B_{G,H})$ to arise from an elliptic decomposition (Definition 4.4.1) $\mathcal{D}^H$ of $V_{\mathbb{R}}$ which involves the following two planes:

$$\mathbb{R}(e_1 + e_1') + \mathbb{R}(e_2 + e_2'), \quad \text{a positive definite plane}$$

$$\mathbb{R}(e_1 - e_1') + \mathbb{R}(e_2 - e_2'), \quad \text{a negative definite plane.}$$

We also assume in the even case that $\mathcal{D}^H$ gives rise to the fixed orientation on $V_{\mathbb{R}}$. As the notation suggests, we *do not assume* that $\mathcal{D}^H$ is independent of H. Recall that H is of the form $H = H^+ \times H^- = \mathrm{SO}(V^+) \times \mathrm{SO}(V^-)$. We define j by choosing elliptic decompositions $\mathcal{D}_H$ of $V^{\pm}$, which are assumed to be compatible with the given orientations in the even case. Then we define j using the trivializations of $T_{\mathcal{D}^H} \xrightarrow{\sim} \mathrm{U}(1)^m$ and $T_{\mathcal{D}_H} \xrightarrow{\sim} \mathrm{U}(1)^{m^-} \times \mathrm{U}(1)^{m^+}$ and the isomorphism

$$\mathrm{U}(1)^{m^-} \times \mathrm{U}(1)^{m^+} \xrightarrow{\sim} \mathrm{U}(1)^m$$

$$((z_1, \cdots, z_{m^-}), (w_1, \cdots, w_{m^+})) \mapsto (z_1, \cdots, z_{m^-}, w_1, \cdots, w_{m^+}).$$

In the odd case, we assume that $\mathcal{D}^H$ is convenient, in the sense of Definition 4.5.12. This means that the negative definite plane $\mathbb{R}(e_1 - e_1') + \mathbb{R}(e_2 - e_2')$ is the last member of $\mathcal{D}^H$. In particular, the factor $\mathrm{U}(1)$ of $T_{\mathcal{D}^H}$ corresponding to this plane is sent under j^{-1} into $\mathrm{SO}(V^+) \subset H$.

In the even case, unless $m = d/2$ is odd and $m^+ = 1$, we assume that $\mathcal{D}^H$ is convenient in the sense of Definition 4.6.11 and Definition 4.7.10. Then the aforementioned factor $\mathrm{U}(1)$ of $T_{\mathcal{D}^H}$ is sent under j^{-1} into $\mathrm{SO}(V^+) \subset H$.

In the even case with m odd and $m^+ = 1$, we assume that the negative definite plane $\mathbb{R}(e_1 - e_1') + \mathbb{R}(e_2 - e_2')$ is the last member of $\mathcal{D}^H$. Then the corresponding factor $\mathrm{U}(1)$ of $T_{\mathcal{D}^H}$ is again sent under j^{-1} into $\mathrm{SO}(V^+) \subset H$. In this case $\mathcal{D}^H$ is not convenient in the sense of Definition 4.6.11, but differs from a convenient e.d. by applying the element $(m-1, m) \in \mathfrak{S}_m$.

6.4.2 Definition of f_∞^H

The fixed choice of $(j, B_{G,H})$ determines a Borel B_H of $H_\mathbb{C}$ containing T_H, a subgroup Ω_* of $\Omega = \Omega(G_\mathbb{C}, T_{G,\mathbb{C}})$, and a bijection

$$\Omega_H \times \Omega_* \to \Omega,$$

where $\Omega_H := \Omega(H_\mathbb{C}, T_{H,\mathbb{C}})$.

Let $\mathbb{V}^*$ be the contragredient representation of $\mathbb{V}$. Let $\varphi_{\mathbb{V}^*}$ be the elliptic Langlands parameter of $G_\mathbb{R}$ corresponding to $\mathbb{V}^*$ (whose L-packet is the packet of discrete series representations of $G(\mathbb{R})$ corresponding to $\mathbb{V}^*$.) Let $\Phi_H(\phi_{\mathbb{V}^*})$ be the set of equivalence classes of elliptic Langlands parameters of $H_\mathbb{R}$ that induces $\varphi_{\mathbb{V}^*}$ via $\eta : {}^LH \to {}^LG$. We have a map

$$\Phi_H(\varphi_{\mathbb{V}^*}) \to \Omega_*$$

$$\varphi_H \mapsto \omega_*(\varphi_H),$$

characterized by the condition that φ_H is aligned with $(\omega_*(\varphi_H)^{-1} \circ j, B_{G,H}, B_H)$.

For any $\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})$, define

$$f_{\varphi_H} := d(H)^{-1} \sum_{\pi \in \Pi(\varphi_H)} f_\pi,$$

where

- the summation is over the discrete series representations π of $H(\mathbb{R})$ inside the L-packet $\Pi(\varphi_H)$ of φ_H.

- For each π, the function $f_\pi \in C_c^\infty(H(\mathbb{R}))$ is a pseudo-coefficient for π. (cf. [11]). Note that this notion depends on the choice of a Haar measure on $H(\mathbb{R})$. We use the one fixed at the beginning of §6.4.

- $d(H) = |\Pi(\phi_H)|$. Note that this number is an invariant of $H_{\mathbb{R}}$, equal to the cardinality of the absolute Weyl group divided by the cardinality of the relative Weyl group of an elliptic maximal torus (over $\mathbb{R}$).

Definition 6.4.3.

$$f_\infty^H := (-1)^{q(G_{\mathbb{R}})} \langle \mu_{T_G}, s \rangle_j \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) f_{\varphi_H}.$$

Here μ_{T_G} is the Hodge cocharacter of any h in the Shimura datum X *that factors through* T_G, and s is in the endoscopic datum. The pairing $\langle \mu_{T_G}, s \rangle_j$ is defined as follows: First we use j to view μ_{T_G} as a cocharacter of T_H. Then μ_{T_G} can be paired with s, which is viewed as an element of $Z(\hat{H})$, via the canonical pairing

$$X_*(T_H) \times Z(\hat{H}) \to \pi_1(H) \times Z(\hat{H}) = X^*(Z(\hat{H})) \times Z(\hat{H}) \to \mathbb{C}^\times.$$

Remark 6.4.4. From how we have chosen j, and the assumption that $s \neq -1$ (i.e. the trivial endoscopic datum is represented by the trivial representative), we have

$$\langle \mu_{T_G}, s \rangle_j = 1.$$

Remark 6.4.5. By construction f_∞^H is stable cuspidal.

6.4.6 Normalizing the transfer factors

We normalize the transfer factors between

$$(H, {}^L H, s, \eta)$$

and G at various places as follows:

We use the canonical unramified normalization for the transfer factor at p (cf. [20]) and use $\Delta_{j,B_{G,H}}$ for the transfer factor at ∞, and normalize the transfer factors away from p and ∞ such that at almost all unramified places we have the canonical unramified normalization and such that the global product formula is satisfied ([53, §6]).

Suppose M is a standard Levi subgroup of G and $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$ inducing $(H, s, \eta) \in \mathscr{E}(G)$ and $(M', s'_M, \eta_M) \in \mathscr{E}(M)$. Again we assume H is cuspidal and unramified at p. It follows that M' is unramified at p, and the endoscopic datum (M', s'_M, η_M) for M is unramified at p. We normalize the transfer factors between $(M', {}^L M, s'_M, \eta_M)$ and M as follows:

Away from p and ∞, we normalize the transfer factors by inheriting the transfer factors between G and H chosen above, in view of the natural identification of M' with a Levi subgroup of H. cf. Remark 6.4.7 below. At p, we use the canonical unramified normalization, which is also the same as that inherited from the canonical unramified normalization between H and G. At ∞, we do not fix an a priori normalization. In fact, a comparison between different normalizations in this case is key to the computation.

Remark 6.4.7. At any place of $\mathbb{Q}$, there is the notion of the normalization of the transfer factor between (M', s'_M, η_M) and M inherited from the normalization of the transfer factor between H and G. It is described using a simple formula as in [59, §5.2] or [60, §5.1]. Roughly speaking, this means that apart from the difference in Δ_{IV}, the transfer factor between M' and M is *equal* to the transfer factor between H and G. Obviously, this notion depends not just on $(M', s'_M, \eta_M) \in \mathscr{E}(G)$, but in fact on $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$, because at least $(H, s, \eta) \in \mathscr{E}(G)$ already depends on the latter. The most important property of this notion is that if the normalizations between H and G at all the places satisfy the global product formula, then so do the inherited normalizations between M' and M at all the places. The validity of this notion is claimed in Kottwitz's unpublished notes but unfortunately it seems that there is no proof in the published literature. As indicated by Kottwitz, this is an easy consequence of the definition in [53]. It can be proved similarly as [20, Lemma 9.2]. Alternatively, in our specific situation, a proof also easily follows from the explicit formulas for the transfer factors in [81].

6.4.8 Definition of f_p^H and $f^{H,\infty}$

Definition 6.4.9. Define $f^{H,p,\infty}$ to be a Langlands-Shelstad transfer of $f^{p,\infty}$ as in Proposition 6.1.1 with respect to the fixed Haar measures on $G(\mathbb{A}_f^p)$ and $H(\mathbb{A}_f^p)$. Define $f_p^H \in \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p))$ to be the object denoted by f^H in Definition 5.4.1.

Remark 6.4.10. Once an element $f_p^H \in \mathcal{H}^{\mathrm{ur}}(H(\mathbb{Q}_p))$ is specified, it still corresponds ambiguously to different functions on $H(\mathbb{Q}_p)$. Namely, for each choice of a hyperspecial subgroup $K_{H,p}$ of $H(\mathbb{Q}_p)$ there is a resulting $K_{H,p}$-bi-invariant function. These functions potentially have different orbital integrals, but they have the same stable orbital integrals, as notes in [40, §7]. The same remark applies to the various canonical constant terms (cf. Corollary 5.2.7) $(f_p^H)_{M'} \in \mathcal{H}^{\mathrm{ur}}(M'(\mathbb{Q}_p))$ for Levi subgroups M' of H defined over $\mathbb{Q}_p$. In particular, the value of the simplified geometric side of the stable trace formula ST^H, defined in §6.3, is well defined when evaluated at the test function $f^H = f_\infty^H f_p^H f^{H,p,\infty}$.

Remark 6.4.11. The definition of f_∞^H depends on the choice of a Haar measure on $H(\mathbb{R})$. The definition of $f^{H,p,\infty}$ depends on Haar measures on $H(\mathbb{A}_f^p)$ and $G(\mathbb{A}_f^p)$. The product function f^H depends only on the Haar measure on $G(\mathbb{A}_f^p)$, since we assume that the product

Haar measure on $H(\mathbb{A})$ (with the factor at p determined by the condition that hyperspecial subgroups have measure 1) is the Tamagawa measure.

Proposition 6.4.12. *Let M be a standard Levi subgroup of G, and let $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$. Let $(H, s, \eta) \in \mathscr{E}(G)$ be the induced endoscopic datum for G, and let $(M', s'_M, \eta_M) \in \mathscr{E}(M)$ be the underlying endoscopic datum for M. Then M' is naturally identified with a Levi subgroup of H, so we may take constant terms of functions on H along M. The function $(f^{H,p,\infty})_{M'} \in C_c^\infty(M'(\mathbb{A}_f^p))$ is a Langlands-Shelstad transfer of $(f^{p,\infty})_M \in C_c^\infty(M(\mathbb{A}_f^p))$ in the sense of Proposition 6.1.1, with respect to the normalization of the transfer factors away from p and ∞ between (M', s'_M, η_M) and M as fixed in §6.4.6.*

Proof. This can be proved similarly as [59, Lemma 6.3.4], which follows Kottwitz's unpublished notes. The only modifications needed are:

- We replace G_γ and M_γ in the argument by G_γ^0 and M_γ^0.
- When citing [48] to deduce that a weak transfer in the sense of Remark 6.1.3 is a transfer in the sense of Proposition 6.1.1, we use the arguments in Remark 6.1.3 to reduce to the case where the result of [48] applies.

□

6.5 Statement of the main computation

We fix the following data:

- a Haar measure dg^p on $G(\mathbb{A}_f^p)$.
- an integer $a \geq 1$.
- an irreducible algebraic representation $\mathbb{V}$ of G defined over a number field L, as in §1.6. We will freely view $\mathbb{V}$ also as a representation of $G_{\mathbb{C}}$.

- a prime number $p > 2$ at which G is unramified.
- a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$.
- a function $f^{p,\infty} \in C_c^\infty(G(\mathbb{A}_f^p))$.

Then as in §6.4, for any elliptic endoscopic datum $(H, {}^L H, s, \eta) \in \mathscr{E}(G)$ (with $s \neq -1$) we obtain a function $f^H = f_\infty^H f^{H,p,\infty} f_p^H$ on $H(\mathbb{A})$. On the other hand, for any standard Levi subgroup M of G, we obtain $\mathrm{Tr}_M(f^{p,\infty} dg^p, a)$ as in Definition 2.6.1. We shall abbreviate the last object as $\mathrm{Tr}_M(\mathbb{V})$.

Definition 6.5.1. Let M be a standard Levi subgroup of G. Define

$$\mathrm{Tr}'_M(\mathbb{V}) = (n_M^G)^{-1} \sum_{(M', s_M, \eta_M) \in \mathscr{E}_G(M)} |\mathrm{Out}_G(M', s_M, \eta_M)|^{-1} \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H),$$

where for each $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$, we write H for the induced endoscopic datum for G. Here we also avoid the representative (M', s_M, η_M) with $s_M = -1$, and as a consequence (the explicit representative of) the induced endoscopic datum H for G satisfies $s \neq -1$.

Recall from Definition 4.11.12 that we define ${}^{\mathrm{a}}\mathbb{V}$ to be the representation obtained by precomposing the representation $\mathbb{V}$ by the automorphism $\mathrm{Int}\, n_{12}$ of G, where n_{12} is defined in Definition 2.3.2. In the even case $n_{12} \in \mathrm{O}(V) - \mathrm{SO}(V)$ and $\mathrm{Int}\, n_{12}$ is not an inner automorphism of G. We now state the main computation as follows:

Theorem 6.5.2. *In the odd case,* $\mathrm{Tr}_M(\mathbb{V}) = \mathrm{Tr}'_M(\mathbb{V})$ *for a large enough. In the even case,* $\mathrm{Tr}_M(\mathbb{V}) + \mathrm{Tr}_M({}^{\mathrm{a}}\mathbb{V}) = \mathrm{Tr}'_M(\mathbb{V}) + \mathrm{Tr}'_M({}^{\mathrm{a}}\mathbb{V})$ *for a large enough.*

In order to simplify notation, it is convenient to allow $\mathbb{V}$ to be a direct sum, or more generally a formal sum [21] of irreducible representations. Thus in the odd case we still

[21] since both Tr_M and Tr'_M only depend on the image of $\mathbb{V}$ in the Grothendieck group.

assume that it is irreducible, but in the even case we assume that it is a formal sum of the form $\mathbb{V}_1 + {}^{\mathrm{a}}\mathbb{V}_1$ for an irreducible representation $\mathbb{V}_1$. In the even case, we define

$$\mathrm{Tr}'_M(\mathbb{V}) := \mathrm{Tr}'_M(\mathbb{V}_1) + \mathrm{Tr}'_M({}^{\mathrm{a}}\mathbb{V}_1),$$

and

$$\mathrm{Tr}_M(\mathbb{V}) := \mathrm{Tr}_M(\mathbb{V}_1) + \mathrm{Tr}_M({}^{\mathrm{a}}\mathbb{V}_1).$$

Note that $\mathrm{Tr}'_M(\mathbb{V})$ can be directly defined similarly as before, with the only modification being that now each function f_∞^H is a sum of two functions, defined using $\mathbb{V}_1$ and ${}^{\mathrm{a}}\mathbb{V}_1$ respectively. Also $\mathrm{Tr}_M(\mathbb{V})$ can be directly defined similarly as before, with the only modification that now each Kostant-Weyl term $L_M(\gamma)$ is a sum of two Kostant-Weyl terms, one for $\mathbb{V}_1$ and one for ${}^{\mathrm{a}}\mathbb{V}_1$.

Thus we may rewrite Theorem 6.5.2 as follows.

Theorem 6.5.3. *Keep the above assumptions on $\mathbb{V}$. Let M be a standard Levi subgroup of G. We have $\mathrm{Tr}_M(\mathbb{V}) = \mathrm{Tr}'_M(\mathbb{V})$ for a large enough.*

In the following we further abbreviate $\mathrm{Tr}_M := \mathrm{Tr}_M(\mathbb{V})$ and $\mathrm{Tr}'_M := \mathrm{Tr}'_M(\mathbb{V})$, where $\mathbb{V}$ is as in Theorem 6.5.3. We prove Theorem 6.5.3 in the following, from §6.6 until §6.14.

6.6 First simplifications

Definition 6.6.1. Denote by $\mathscr{E}(M)^c$ (resp. $\mathscr{E}(G)^c$) the subset of $\mathscr{E}(M)$ (resp. $\mathscr{E}(G)^c$) consisting of endoscopic data whose groups are cuspidal and unramified at p. Denote by $\mathscr{E}_G(M)^c$ the subset of $\mathscr{E}_G(M)$ consisting of elements such that their images in $\mathscr{E}(G)$ are in $\mathscr{E}(G)^c$ and their images in $\mathscr{E}(M)$ are in $\mathscr{E}(M)^c$.

Remark 6.6.2. In the odd case, $\mathscr{E}(M) = \mathscr{E}(M)^c, \mathscr{E}(G) = \mathscr{E}(G)^c, \mathscr{E}_G(M) = \mathscr{E}_G(M)^c$.

6.6.3

In Definition 6.5.1, we may restrict the range of the summation to

$$(M', s_M, \eta_M) \in \mathscr{E}_G(M)^c,$$

because all the other terms are by definition automatically zero. Also note that in the even case and $M = M_2$, both sides of Theorem 6.5.3 are automatically zero because M_2 is not cuspidal. (cf. Remark 2.6.2.)

Lemma 6.6.4. *$n^G_{M_{12}} = 8,\ n^G_{M_1} = n^G_{M_2} = 2.$*

Proof. Direct computation. □

Lemma 6.6.5.

$$n^G_M \operatorname{Tr}'_M = \sum_{M' \in \mathscr{E}(M)^c} |\mathrm{Out}_M(M')|^{-1} \sum_A \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H).$$

Here the summation over A is for A running through the following sets:

- *In the odd case for $M = M_{12}$, $A \in \mathscr{P}(\{1,2\})$.*
- *In the even case for $M = M_{12}$, $A \in \{\emptyset, \{1,2\}\}$.*
- *For $M = M_1$, $A \in \{\emptyset, \{1,2\}\}$.*
- *In the odd case for $M = M_2$, $A \in \{\emptyset, \{1\}\}$.*

In the above summation, whenever $M' \in \mathscr{E}(M)$ and A are fixed, we use them to get an element in $\mathscr{E}_G(M)$, from which we define $H \in \mathscr{E}(G)$. (Recall that the set $\mathscr{E}_G(M)$ is parametrized by suitable combinations of a choice of A and a choice of a parameter (d^+, d^-) or $(d^+, \delta^+, d^-, \delta^-)$ for the set $\mathscr{E}(M)$.) We also assume that the summation over $M' \in \mathscr{E}(M)^c$ is *over the explicitly chosen representatives and the representative* (M', s'_M, η_M) *with* $s'_M = -1$ *is not used.*

Proof. To unify notation, in the odd case we shall sometimes write $\mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(G, M)$ for elements in $\mathscr{E}_G(M)$ and write $\mathfrak{e}_{d^+,\delta^+,d^-,\delta^-}(M)$ for elements in $\mathscr{E}(M)$, understanding that $\delta^\pm = 1$. We also write $(A, d^+, \delta^+, d^-, \delta^-) := \mathfrak{e}_{A,d^+,\delta^+,d^-,\delta^-}(G, M)$ and $(d^+, \delta+, d^-, \delta^-) := \mathfrak{e}_{d^+,\delta+,d^-,\delta^-}(M)$.

We turn the summation in Definition 6.5.1 over $\mathscr{E}_G(M)^c$ (cf. §6.6.3) into a double summation: the summation over $\mathscr{E}(M)^c$ of the summations over the fibers of the natural forgetful map

$$\mathscr{E}_G(M)^c \to \mathscr{E}(M)^c \tag{6.6.1}$$

$$(A, d^+, \delta^+, d^-, \delta^-) \text{ up to simultaneous swapping} \mapsto (d^+, \delta^+, d^-, \delta^-) \text{ up to swapping.}$$

Odd case M_{12}:

When $d^+ \neq d^-$, the fiber of (6.6.1) over (d^+, d^-) is equal to $\mathscr{P}(\{1, 2\})$ and has 4 elements. When $d^+ = d^-$, the fiber over (d^+, d^-) is equal to $\mathscr{P}(\{1, 2\})/(A \sim A^c)$, and has 2 elements. In this case we will still sum over $\mathscr{P}(\{1, 2\})$, and introduce a factor 1/2 in the summation. Note that $\mathrm{Out}_G(\mathfrak{e}_{A,d^+,d^-})$ is trivial and $\mathrm{Out}_M(\mathfrak{e}_{d^+,d^-}(M))$ is trivial unless $d^+ = d^-$, in which case it has order 2. The lemma follows.

Even case M_{12}:

When $(d^+, \delta^+) \neq (d^-, \delta^-)$, the fiber over $(d^+, \delta^+, d^-, \delta^-)$ of (6.6.1) is equal to $\{\emptyset, \{1, 2\}\}$, because the choices $A = \{1\}$ or $\{2\}$ violate the cuspidality condition. When $(d^+, \delta^+) = (d^-, \delta^-)$, the fiber over $(d^+, \delta^+, d^-, \delta^-)$ is a singleton. In this case we will still sum over $\{\{1, 2\}, \emptyset\}$, and introduce a factor 1/2 in the summation.

Note that $|\mathrm{Out}_G(M', s_M, \eta_M)| = 2$ for all $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$ such that $M' \neq M^*$, and in the rest cases $|\mathrm{Out}_G(M', s_M, \eta_M)| = 1$. Therefore, when $A \in \{\emptyset, \{1,2\}\}$, we have

$$\left|\mathrm{Out}_M(d^+, \delta^+, d^-, \delta^-)\right| / \left|\mathrm{Out}_G(A, d^+, \delta^+, d^-, \delta^-)\right| = \begin{cases} 1, & (d^+, \delta^+) \neq (d^-, \delta^-) \\ 2, & (d^+, \delta^+) = (d^-, \delta^-) \end{cases}$$

The lemma follows.

Case M_1:

When $(d^+, \delta^+) \neq (d^-, \delta^-)$, the fiber of (6.6.1) over $(d^+, \delta^+, d^-, \delta^-)$ is equal to $\{\emptyset, \{1,2\}\}$. Other wise the fiber is a singleton. In this case we will still sum over $\{\{1,2\}, \emptyset\}$, and introduce a factor $1/2$.

Note that we have

$$\left|\mathrm{Out}_M(d^+, \delta^+, d^-, \delta^-)\right| / \left|\mathrm{Out}_G(A, d^+, \delta^+, d^-, \delta^-)\right| = \begin{cases} 1, & (d^+, \delta^+) \neq (d^-, \delta^-) \\ 2, & (d^+, \delta^+) = (d^-, \delta^-) \end{cases}$$

The lemma follows.

Odd case M_2:

When $d^+ \neq d^-$, the fiber of (6.6.1) over (d^+, d^-) is equal to $\mathscr{P}(\{1\}) = \{\emptyset, \{1\}\}$. When $d^+ = d^-$, the fiber over (d^+, d^-) is a singleton. In this case we will still sum over $\mathscr{P}(\{1\})$, and introduce a factor $1/2$.

Note that $\mathrm{Out}_G(\mathfrak{e}_{A,d^+,d^-})$ is trivial and $\mathrm{Out}_M(\mathfrak{e}_{d^+,d^-}(M))$ is trivial unless $d^+ = d^-$, in which case it has order 2. The lemma follows.

$\square$

6.7 Computation of $ST^H_{M'}(f^H)$

We study the terms $ST^H_{M'}(f^H)$ in Lemma 6.6.5.

Definition 6.7.1. Let $\Sigma(M')$ be a set of representatives in $M'(\mathbb{Q})$ of the stable conjugacy classes in $M'(\mathbb{Q})$ that are $\mathbb{R}$-elliptic. When $M = M_1$, so that $M^{\mathrm{GL}} = (M')^{\mathrm{GL}} = \mathrm{GL}_2$, we assume that each element $\gamma' \in \Sigma(M')$ satisfies the condition that its component in $(M')^{\mathrm{GL}}(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$, which by assumption lies in an elliptic maximal torus of $\mathrm{GL}_{2,\mathbb{R}}$, lies in the standard elliptic maximal torus

$$T^{\mathrm{std}}_{\mathrm{GL}_2} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\}.$$

By definition, each term $ST^H_{M'}(f^H)$ in Lemma 6.6.5 has an expansion:

$$ST^H_{M'}(f^H) := \tau(M') \sum_{\gamma' \in \Sigma(M')} \bar{\iota}^{M'}(\gamma')^{-1} SO_{\gamma'}(f^{H,\infty}_{M'}) S\Phi^H_{M'}(\gamma', f^H_\infty). \tag{6.7.1}$$

Lemma 6.7.2. *In (6.7.1), only those γ' that are (M, M')-regular contribute non-trivially.*

Proof. If X, Y are two quantities we write $X \sim Y$ to mean that "$Y = 0 \Rightarrow X = 0$". Let γ' be any semi-simple elliptic element of $M'(\mathbb{R})$. We have

$$S\Phi^H_{M'}(\gamma', f^H_\infty) \sim \sum_{\Pi \text{ for H}} \Phi^H_{M'}(\gamma'^{-1}, \Theta_\Pi) \operatorname{Tr}(f^H_\infty, \Pi) \sim$$

$$\sim \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H)) \Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}),$$

where the summation $\sum_{\Pi \text{ for H}}$ is over the discrete series L-packets for $H_{\mathbb{R}}$. By [60, Proposition 3.2.5, Remarque 3.2.6], the last quantity above is zero provided that γ' is not (M, M')-regular. $\square$

Remark 6.7.3. In the later stage of the proof of Theorem 6.5.3 we will use the quantitative statement in [60, Proposition 3.2.5].

We continue the study of (6.7.1). By Lemma 6.7.2, we only need to sum over those $\gamma' \in \Sigma(M')$ that are (M, M')-regular. By Proposition 6.4.12, we may further restrict to those γ' that is an image of a semi-simple element $\gamma_M \in M(\mathbb{A}_f^p)$ (in the adelic sense), and in this case we have

$$SO_{\gamma'}(f_{M'}^{H,p,\infty}) = (\Delta_{M'}^M)_A^{p,\infty}(\gamma', \gamma_M) O_{\gamma_M}^{s'_M}(f_M^{p,\infty}), \tag{6.7.2}$$

where s'_M is as in the endoscopic datum $(M', s'_M, \eta_M) \in \mathscr{E}(M)$. We remind the reader that it is different from s_M as in $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$ as the latter depends on A. The transfer factor $(\Delta_{M'}^M)_A^{p,\infty}$ is the one inherited from Δ_H^G (cf. Remark 6.4.7), which indeed depends on $(M', s_M, \eta_M) \in \mathscr{E}_G(M)$ (i.e. it depends on A).

Definition 6.7.4. Let $\epsilon^{p,\infty}(A) \in \mathbb{C}^\times$ be the constant such that

$$(\Delta_{M'}^M)_A^{p,\infty} = \epsilon^{p,\infty}(A)(\Delta_{M'}^M)_\emptyset^{p,\infty}.$$

Definition 6.7.5. Define $\Sigma(M')_1$ to be the subset of $\Sigma(M')$ consisting of elements γ' such that γ' is (M, M')-regular and is an image of a semi-simple element in $M(\mathbb{A}_f^p)$.

Definition 6.7.6. For a fixed $\mathfrak{e} = (M', s_M, \eta_M) \in \mathscr{E}(M)$, and a fixed $\gamma' \in \Sigma(M')_1$, let $\gamma_M \in M(\mathbb{A}_f^p)$ be a semi-simple element of whom γ' is an image, and define

$$I(\mathfrak{e}, \gamma') := \bar{\iota}^{M'}(\gamma')^{-1}(\Delta_{M'}^M)_\emptyset^{p,\infty}(\gamma', \gamma_M) O_{\gamma_M}^{s'_M}(f_M^{p,\infty}) \times$$
$$\times \sum_A \epsilon^{p,\infty}(A)\tau(G)\tau(H)^{-1}\tau(M') SO_{\gamma'}(f_{p,M'}^H) S\Phi_{M'}^H(\gamma', f_\infty^H)$$

where $\sum_A$ is the same as Lemma 6.6.5. By (6.7.2) this definition is independent of the choice of γ_M.

Lemma 6.7.7. *We have*

$$n_M^G \operatorname{Tr}'_M = \sum_{\mathfrak{e}\in\mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'\in\Sigma(M')_1} I(\mathfrak{e},\gamma').$$

Proof. Direct consequence of Lemma 6.6.5 and the above discussion. □

Definition 6.7.8. For $\mathfrak{e}$ and γ' as in Definition 6.7.6, and A as in Lemma 6.6.5, we define

$$K(\mathfrak{e},\gamma',A) := (-1)^{q(G_{\mathbb{R}})} \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}).$$

Lemma 6.7.9. *For $\mathfrak{e}$ and γ' as in Definition 6.7.6, we have*

$$I(\mathfrak{e},\gamma') = \bar{\iota}^{M'}(\gamma')^{-1}(\Delta^M_{M'})^{p,\infty}_{\emptyset}(\gamma',\gamma_M)O^{s'_M}_{\gamma_M}(f^{p,\infty}_M)\tau(M)k(M)k(G)^{-1}(-1)^{\dim A_{M'}}\bar{v}(M'^0_{\gamma'})^{-1}\times$$

$$\times\sum_A \epsilon^{p,\infty}(A)SO_{\gamma'}(f^H_{p,M'})K(\mathfrak{e},\gamma',A).$$

Here $\sum_A$ is the same as in Lemma 6.6.5.

Proof. Direct consequence of (6.7.6), Definition 6.4.3, Remark 6.4.4, Corollary 6.2.5. □

6.8 Computation of K

We use the notation of Lemma 6.7.9 and focus our attention to one term

$$K(\mathfrak{e},\gamma',A) := (-1)^{q(G_{\mathbb{R}})} \sum_{\varphi_H\in\Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1},\Theta_{\varphi_H}).$$

We would like to apply [60, Proposition 3.2.5] to compute the above summation. First we need some preparations.

Fix an elliptic maximal torus $T_{M'}$ of $M'_{\mathbb{R}}$, such that $\gamma' \in T_{M'}(\mathbb{R})$. Then $T_{M'}$ is of the form

$$T_{M'} = T_{(M')^{\mathrm{GL}}} \times T_{(M')^{\mathrm{SO},-}} \times T_{(M')^{\mathrm{SO},+}}$$

where $T_{(M')^{\mathrm{GL}}}$ (resp. $T_{(M')^{\mathrm{SO},-}}$, resp. $T_{(M')^{\mathrm{SO},+}}$) is an elliptic maximal torus of $(M')^{\mathrm{GL}}_{\mathbb{R}}$ (resp. $(M')^{\mathrm{SO},-}_{\mathbb{R}}$, resp. $(M')^{\mathrm{SO},+}_{\mathbb{R}}$). When $M = M_{12}$ or when in the odd case $M = M_2$, we have necessarily $T_{(M')^{\mathrm{GL}}} = (M')^{\mathrm{GL}}$. When $M = M_1$, by our assumption on γ' in Definition 6.7.1, we know that we may assume $T_{(M')^{\mathrm{GL}}} = T^{\mathrm{std}}_{\mathrm{GL}_2}$. We assume this, which shall have important consequences in §6.10.

We then fix an elliptic maximal torus T_M of $M_{\mathbb{R}}$, and an admissible isomorphism

$$j_M : T_{M'} \xrightarrow{\sim} T_M.$$

We may and will assume that T_M is of the form

$$T_M = T_{(M)^{\mathrm{GL}}} \times T_{M^{\mathrm{SO}}},$$

where $T_{M^{\mathrm{GL}}} = T_{(M')^{\mathrm{GL}}}$, and $T_{M^{\mathrm{SO}}}$ is an elliptic maximal torus of M^{SO}. We also assume that j_M is the product of the identity on $T_{(M')^{\mathrm{GL}}}$ and an admissible isomorphism

$$T_{(M')^{\mathrm{SO},-}} \times T_{(M')^{\mathrm{SO},+}} \xrightarrow{\sim} T_{M^{\mathrm{SO}}},$$

in the sense that $(M')^{\mathrm{SO}}$ (and the additional datum induced by $\mathfrak{e} = (M', s'_M, \eta_M)$) is naturally an endoscopic datum for M^{SO}.

For any choice of a Borel B_0 of $G_{\mathbb{C}}$ containing $T_{M,\mathbb{C}}$, we get a unique isomorphism

$$\Upsilon_{B_0} : T_{M,\mathbb{C}} \xrightarrow{\sim} \mathbb{G}_m^m,$$

characterized by the condition that under Υ_{B_0}, the based root system on $X^*(T_M)$ determined by B_0 is transported to the standard based root system on $X^*(\mathbb{G}_m^m)$ of type B or D. Under $\Upsilon_{B_0}^{-1}$, the m copies of $\mathbb{G}_m$ define an ordered m-tuple of trivialized (i.e. with an isomorphism to $\mathbb{G}_m$) subgroups of $T_{M,\mathbb{C}}$. We name them by

$$\tau_{0_1}, \tau_{0_2}, \tau_1, \tau_2, \cdots, \tau_{m-2} \text{ if } M = M_{12} \text{ or } M_1,$$

and by

$$\tau_0, \tau_1, \cdots, \tau_{m-1} \text{ if } M = M_2 \text{ (odd case)}.$$

Thus for example when $M = M_{12}$, Υ_{B_0} maps τ_{0_2} to the subgroup $\{(1, z, 1, \cdots, 1) | z \in \mathbb{G}_m\}$ of $\mathbb{G}_m^m$. We fix a choice of B_0, depending on the choice of j_M, satisfying the following conditions.

1. When $M = M_{12}$, we require that τ_{0_1} and τ_{0_2} are respectively the first $\mathbb{G}_m$ (i.e. $\mathrm{GL}(V_1)$) and the second $\mathbb{G}_m$ (i.e. $\mathrm{GL}(V_2/V_1)$) of $T_M^{\mathrm{GL}} = M^{\mathrm{GL}}$. When $M = M_1$, we require that $\tau_{0_1} \times \tau_{0_2}$ is equal to $T_{M^{\mathrm{GL}},\mathbb{C}}$. When $M = M_2$, we require that τ_0 is equal to $T_{M^{\mathrm{GL}}} = M_{\mathrm{GL}} = \mathrm{GL}(V_1)$.

2. $j_M^{-1}(\tau_1 \times \tau_2 \times \cdots \times \tau_{m^-}) = T_{(M')^{\mathrm{SO},-},\mathbb{C}}$.

Remark 6.8.1. We emphasize that we choose $(T_M, T_{M'}, j_M)$ only depending on $(\mathfrak{e}, \gamma')$, and independently of A.

Up to now the discussion has not involved A. We now take A into account, so we have an endoscopic datum (H, s, η) for G induced by $(\mathfrak{e}, A)$. Recall from §6.4.1 that we have fixed $(T_H, T_G, j, B_{G,H})$. Similarly as above, the pair $(T_G, B_{G,H})$ defines an ordered m-tuple of trivialized subgroups of $T_{G,\mathbb{C}}$. We name them respectively by

$$\rho_1, \rho_2, \cdots, \rho_m.$$

By the construction in §6.4.1, we know that

$$j^{-1}(\rho_1 \times \rho_2 \times \cdots \times \rho_{m^-}) \subset H^-, \quad j^{-1}(\rho_{m^-+1} \times \rho_{m^-+2} \times \cdots \times \rho_m) \subset H^+.$$

Definition 6.8.2. Define an isomorphism $i_G(A) : T_{M,\mathbb{C}} \xrightarrow{\sim} T_{G,\mathbb{C}}$ as follows. When $M = M_{12}$ or M_1, let $i_G(A)$ map $\tau_{0_1}, \tau_{0_2}, \tau_1, \cdots, \tau_{m-2}$ respectively to

$$\begin{cases} \rho_1, \rho_2, \cdots, \rho_m, & A = \emptyset \\ \rho_{m^-+1}, \rho_1, \rho_2, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{1\} \\ \rho_1, \rho_{m^-+1}, \rho_2, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{2\} \\ \rho_{m^-+1}, \rho_{m^-+2}, \rho_1, \cdots, \rho_{m^-}, \rho_{m^-+3}, \cdots, \rho_m, & A = \{1, 2\} \end{cases}$$

(When $M = M_1$ or in the even case when $M = M_{12}$, the parameter A can only assume $\{1, 2\}$ and $\emptyset$, but we still use the above formula).

When $M = M_2$ (in the odd case), let $i_G(A)$ map $\tau_0, \tau_1, \cdots, \tau_{m-1}$ respectively to

$$\begin{cases} \rho_1, \cdots, \rho_m, & A = \emptyset \\ \rho_{m^-+1}, \rho_1, \cdots, \rho_{m^-}, \rho_{m^-+2}, \cdots, \rho_m, & A = \{1\} \end{cases}$$

Definition 6.8.3. Define $i_H(A)$ to be the unique isomorphism $T_{M',\mathbb{C}} \xrightarrow{\sim} T_{H,\mathbb{C}}$ that fits into the following commutative diagram:

$$\begin{tikzcd} T_{H,\mathbb{C}} \arrow[r, "j"] & T_{G,\mathbb{C}} \\ T_{M',\mathbb{C}} \arrow[u, "i_H(A)"] \arrow[r, "j_M"] & T_{M,\mathbb{C}} \arrow[u, "i_G(A)"'] \end{tikzcd}$$

Lemma 6.8.4. *$i_G(A)$ (resp. $i_H(A)$) is induced by an inner automorphism of $G_\mathbb{C}$ (resp. $H_\mathbb{C}$).*

Proof. Obvious from the definition. □

Definition 6.8.5. Define the four Borel subgroups:

- B_M, the Borel of $M_{\mathbb{C}}$ containing $T_{M,\mathbb{C}}$, defined to be $B_0 \cap M$.
- B_G, the Borel of $G_{\mathbb{C}}$ containing $T_{G,\mathbb{C}}$, defined to be $i_G(A)_* B_0$. This can be different from $B_{G,H}$ fixed in §6.4.1.
- $B_{M'}$, the Borel of $M'_{\mathbb{C}}$ containing $T_{M',\mathbb{C}}$, defined to be the one induced by (B_M, j_M).
- B'_H, the Borel of $H_{\mathbb{C}}$ containing $T_{H,\mathbb{C}}$, defined to be $i_H(A)_*(B_{M'})$, or equivalently defined from (B_G, j). This can be different from B_H fixed in §6.4.1.

We recall [60, Proposition 3.2.5] in our notation.

Proposition 6.8.6.

$$\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Delta^A_{j_M,B_M}(\gamma', j_M(\gamma'))\Phi^G_M(j_M(\gamma')^{-1}, \Theta^H_{\varphi_{\mathbb{V}^*}}) = \\ = \sum_{\varphi_H \in \Phi_H(\varphi_{\mathbb{V}^*})} \det(\omega'_*(\varphi_H))\Phi^H_{M'}(\gamma'^{-1}, \Theta_{\varphi_H}) \tag{6.8.1}$$

where ϵ_R is defined using B_M (i.e. it is -1 to the number of B_M-positive roots for $(G_{\mathbb{C}}, T_{M,\mathbb{C}})$ that are real and map $j_M(\gamma'^{-1})$ to $]0,1[$), and ϵ_{R_H} is defined similarly using $B_{M'}$. The elements $\omega'_*(\varphi_H) \in \Omega'_*$ are determined using (j, B_G) in the way similar to that described in §6.4.2 (which used $(j, B_{G,H})$ instead). The transfer factor $\Delta^A_{j_M,B_M}$ is defined as [22]

$$\Delta^A_{j_M.B_M} := (-1)^{q(G_{\mathbb{R}})+q(H_{\mathbb{R}})+q(M_{\mathbb{R}})+q(M'_{\mathbb{R}})}\Delta_{j_M,B_M}.$$

[22] In loc. cit. $\Delta^A_{j_M,B_M}$ is denoted simply by Δ_{j_M,B_M}, but its definition indeed involves the sign $(-1)^{q(G_{\mathbb{R}})+q(H_{\mathbb{R}})}$, instead of $(-1)^{q(M_{\mathbb{R}})+q(M'_{\mathbb{R}})}$.

The term $\Phi^G_M(\cdot,\Theta^H_{\varphi\mathbb{V}^*})$ *is given as follows. When* $M=M_{12}$ *in the odd case,*

$$\Phi^G_M(\cdot,\Theta^H_{\varphi_{\mathbb{V}^*}}):=\begin{cases}\Phi^G_M(\cdot,\Theta_{\mathbb{V}^*}), & A=\{1,2\}\ \ \mathit{or}\ \emptyset\\ \Phi^G_M(\cdot,\Theta_{\mathbb{V}^*})_{\mathrm{endosc}}, & A=\{1\}\ \ \mathit{or}\ \ \{2\}\end{cases}$$

In all the other cases,

$$\Phi^G_M(\cdot,\Theta^H_{\varphi_{\mathbb{V}^*}}):=\Phi^G_M(\cdot,\Theta_{\mathbb{V}^*}).$$

(In the even case the above is understood to be the sum $\Phi^G_M(\cdot,\Theta_{\mathbb{V}^*_1})+\Phi^G_M(\cdot,\Theta_{^a\mathbb{V}^*_1})$*)* □

For a fixed φ_H, we investigate the relation between $\omega_*(\varphi_H)$ and $\omega'_*(\varphi_H)$. Write $\omega_*:=\omega_*(\varphi_H)$ and $\omega'_*:=\omega'_*(\varphi_H)$ in short. By definition, φ_H is aligned with $(\omega_*^{-1}\circ j,B_{G,H},B_H)$ and also aligned with $((\omega'_*)^{-1}\circ j,B_G,B'_H)$. Suppose $\omega_0\in\Omega(G_{\mathbb{C}},T_{G,\mathbb{C}})$ measures the difference between B_G and $B_{G,H}$ in the sense that the map $\hat{T}_G\to\hat{G}$ determined by B_G and $\varphi_{\mathbb{V}^*}$, is equal to the map $\hat{T}_G\to\hat{G}$ determined by $B_{G,H}$ and $\varphi_{\mathbb{V}^*}$, precomposed with ω_0. Write $\omega_0=\omega_{0,H}\omega_{0,*}$ according to the decomposition of $\Omega(G_{\mathbb{C}},T_{G,\mathbb{C}})$ determined by $(j,B_{G,H})$. Then we have

$$\omega'_*=\omega_0^{-1}\omega_*\omega_{0,H}.$$

In particular,

$$\det(\omega'_*)=\det(\omega_*)\det(\omega_{0,*}^{-1}).$$

Understanding that A only assumes the ranges in Lemma 6.6.5, we compute:

(6.8.2)
$$\det(\omega_{0,*})=\begin{cases}1,\ A=\emptyset\\ (-1)^{m^-},\ A=\{1\}\\ (-1)^{m^-+1},\ A=\{2\}\\ 1,\ A=\{1,2\}\end{cases}$$

Corollary 6.8.7. *We have*

$$K(\mathfrak{e},\gamma',A) = (-1)^{q(G_{\mathbb{R}})}\det(\omega_{0,*})\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\times$$

$$\times\Delta^{A}_{j_M,B_M}(\gamma',j_M(\gamma'))\Phi^G_M(j_M(\gamma')^{-1},\Theta^H_{\varphi_{\mathbb{V}^*}}),$$

where $\det(\omega_{o,*})$ *is given in (6.8.2).*

Proof. Consequence of Proposition 6.8.6. □

6.9 Computation of some signs

We continue computing $I(\mathfrak{e},\gamma')$ from Lemma 6.7.9 and Corollary 6.8.7.

Definition 6.9.1. Let $\text{♫}(A)\in\mathbb{C}^{\times}$, be the constant such that

$$(\Delta^M_{M'})^{p,\infty}_A,\ (\Delta^M_{M'})_p,\ \ \text{♫}(A)\Delta^A_{j_M,B_M}$$

satisfy the global product formula. Equivalently, $\text{♫}(A)\Delta^A_{j_M,B_M}$ is inherited from $\Delta_{j,B_{G,H}}$. cf. §6.4.6 and Remark 6.4.7.

Remark 6.9.2. We have

$$\Delta^A_{j_M,B_M}\epsilon^{p,\infty}(A) = \Delta^{\emptyset}_{j_M,B_M}\,\text{♫}(A)\,\text{♫}(\emptyset)^{-1}.$$

Proposition 6.9.3. *For all M, understanding that A only assumes the ranges in Lemma 6.6.5, we have*

$$\text{♫}(\emptyset) = -1.$$

$$♫(A)\,♫(\emptyset)^{-1} = \begin{cases} 1, & A = \emptyset \\ -1 & A = \{1,2\} \\ (-1)^{m^{-}+1}, & A = \{1\} \;\; or \;\; \{2\} \end{cases}$$

Proof. The sign $♫(A)$ is determined by the condition that $♫(A)\Delta^{(H)}_{j_M,B_M}$ is inherited from $\Delta_{j,B_{G,H}}$. In the rest part of the proof we pass to the local notation over $\mathbb{R}$. We use the phrase "Whittaker normalization" or "Whittaker normalized" to mean the notions defined in Definition 4.5.8 and Definition 4.6.8, and the type I Whittaker normalization defined in Definition 4.7.7.

We claim that the Whittaker normalized transfer factor between M' and M and the Whittaker normalized transfer factor between H and G satisfy the condition that the former is inherited from the latter as in Remark 6.4.7. In fact, when d is not divisible by 4 the claim follows from the uniqueness of Whittaker datum for G^* and M^* and the compatibility assumptions (1)(2) in the beginning of §4.8 (around (4.8.1)). When d is divisible by 4, the claim follows from the compatibility assumptions (1)(2) just mentioned and Proposition 4.7.14.

It follows from the above claim that $♫(A)$ is the product of the following three signs:

1. the sign between $\Delta^{(H)}_{j_M,B_M}$ and Δ_{j_M,B_M}, namely $(-1)^{q(G)+q(H)+q(M)+q(M')}$.

2. the sign between Δ_{j_M,B_M} and the Whittaker normalization.

3. the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization.

Denote by $m^{\pm}$ (resp. $n^{\pm}$) the absolute ranks of $(M')^{\mathrm{SO},\pm}$ (resp. $H^{\pm}$). Denote by m the absolute rank of G.

Odd case M_{12}:

We have

$$q(G) = \frac{(2m-1)2}{2} = 2m-1,$$

$$q(H) = \frac{m^+(m^+ + 1) + m^-(m^- + 1)}{2},$$

$$q(M) = 0,$$

$$q(M') = \frac{n^+(n^+ + 1) + n^-(n^- + 1)}{2}.$$

When $A = \emptyset$ or $\{1, 2\}$, we have

$$\begin{cases} m^+ = n^+ \\ m^- = n^- + 2 \end{cases} \quad \text{or} \quad \begin{cases} m^+ = n^+ + 2 \\ m^- = n^- \end{cases}$$

respectively. First assume the first case. Then

$$q(G)+q(H)+q(M)+q(M') = 2m-1+\frac{2m^+(m^+ + 1) + m^-(m^- + 1) + (m^- - 2)(m^- - 1)}{2} \equiv$$

$$\equiv 1 + m^+(m^+ + 1) + \frac{2(m^-)^2 - 2m^- + 2}{2} \equiv$$

$$\equiv 1 + m^+(m^+ + 1) + m^-(m^- - 1) + 1 \equiv 0 \mod 2.$$

Observing symmetry we get the same result for the second case. We conclude that when $A = \emptyset$ or $\{1, 2\}$, the sign $(-1)^{q(G)+q(H)+q(M)+q(M')}$ is 1.

Now assume $A = \{1\}$ or $\{2\}$. Then $m^+ = n^+ + 1, m^- = n^- + 1$. We have

$$q(G) + q(H) + q(M) + q(M') =$$

$$= 2m - 1 + \frac{m^+(m^+ + 1) + m^+(m^+ - 1) + m^-(m^- + 1) + m^-(m^- - 1)}{2} \equiv$$

$$\equiv 1 + \frac{2(m^+)^2 + 2(m^-)^2}{2} \equiv 1 + m^+ + m^- = m + 1 \mod 2.$$

We conclude that

$$(-1)^{q(G)+q(H)+q(M)+q(M')} = \begin{cases} 1, \ A = \{1,2\} \ \text{or} \ \emptyset \\ (-1)^{m+1}, \ A = \{1\} \ \text{or} \ \{2\} \end{cases}.$$

We now compute the sign between Δ_{j_M,B_M} and the Whittaker normalization. Applying the $q = 0$ case of Corollary 4.5.14, this sign is

$$(-1)^{\lceil n^+/2 \rceil}.$$

Also applying the $q = 2$ case of Corollary 4.5.14, we see that the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lceil m^+/2 \rceil + 1}.$$

Thus

$$\text{♫}(A) = (-1)^{q(G)+q(H)+q(M)+q(M')+\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1}.$$

We have

$$\text{♫}(\emptyset) = (-1)^{\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1} = (-1)^{\lceil m^+/2 \rceil + \lceil m^+/2 \rceil + 1} = -1.$$

$$\text{♫}(\{1,2\}) = (-1)^{\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1} = (-1)^{\lceil (m^+-2)/2 \rceil + \lceil m^+/2 \rceil + 1} = 1.$$

$$\text{♫}(\{1\}) = (-1)^{q(G)+q(H)+q(M)+q(M')+\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1} =$$

$$= (-1)^{m+1+\lceil (m^+-1)/2 \rceil + \lceil m^+/2 \rceil + 1} = (-1)^{m+m^+} = (-1)^{m^-}$$

$$\text{♫}(\{2\}) = (-1)^{q(G)+q(H)+q(M)+q(M')+\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1} =$$

$$= (-1)^{m+1+\lceil (m^+-1)/2 \rceil + \lceil m^+/2 \rceil + 1} = (-1)^{m+m^+} = (-1)^{m^-}$$

This finishes the proof in this case.

Even case M_{12}:

Note that $q(G), q(H), q(M), q(M')$ are all even, since all the four real groups are of the form of a torus times a cuspidal even orthogonal group, namely some $\mathrm{SO}(a, b)$ with a, b even, and we have $q(\mathrm{SO}(a, b)) = ab/2$.

When $A = \emptyset$ or $\{1, 2\}$, we have

$$\begin{cases} m^+ = n^+ \\ m^- = n^- + 2 \end{cases} \quad \text{or} \quad \begin{cases} m^+ = n^+ + 2 \\ m^- = n^- \end{cases}$$

respectively.

We now compute the sign between Δ_{j_M, B_M} and the Whittaker normalization. Applying the $q = 0$ case of Corollary 4.6.13 and Corollary 4.7.12, this sign is

$$(-1)^{\lfloor n^-/2 \rfloor}.$$

Assume it is not the case that m is odd and $m^+ = 1$. Then $\mathcal{D}^H$ which was used to define $(j, B_{G,H})$ in §6.4.1 is convenient in the sense of Definition 4.6.11 and Definition 4.7.10. Applying the $q = 2$ case of Corollary 4.6.13 and Corollary 4.7.12, we see that the sign between $\Delta_{j, B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lfloor m^-/2 \rfloor}.$$

Now assume m is odd and $m^+ = 1$. In this case $\mathcal{D}^H$ used to define $(j, B_{G,H})$ differs from a convenient e.d. by the transposition $(m - 1, m) \in \mathfrak{S}_m$. Let $B'_{G,H}$ be the image of $B_{G,H}$ under $(m - 1, m)$, viewed as an element in the complex Weyl group. Similarly as the proof of the second statement of Lemma 4.7.4, it is easy to show that

$$\Delta_{j,B_{G,H}} = \langle a_{(m-1,m)}, s\rangle \Delta_{j,B'_{G,H}} = -\Delta_{j,B'_{G,H}}.$$

Hence the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is -1 times the sign $(-1)^{\lfloor m^-/2\rfloor+1}$ in Corollary 4.6.13. Namely, it is again

$$(-1)^{\lfloor m^-/2\rfloor}.$$

Thus

$$♫(A) = (-1)^{\lfloor n^-/2\rfloor+\lfloor m^-/2\rfloor}.$$

We have

$$♫(\emptyset) = (-1)^{\lfloor (m^- -2)/2\rfloor+\lfloor m^-/2\rfloor} = -1.$$

$$♫(\{1,2\}) = (-1)^{\lfloor m^-/2\rfloor+\lfloor m^-/2\rfloor} = 1.$$

This finishes the proof in this case.

Odd case M_1:

We have

$$q(G) = \frac{(2m-1)2}{2} = 2m-1,$$

$$q(H) = \frac{m^+(m^++1)+m^-(m^-+1)}{2},$$

$$q(M') = q((M')^{\mathrm{SO}}) + q((M')^{\mathrm{GL}}) = \frac{n^+(n^++1)+n^-(n^-+1)}{2} + 1.$$

$$q(M) = q(\mathrm{GL}_2) = 1.$$

When $A = \emptyset$ or $\{1,2\}$, we have

$$\begin{cases} m^+ = n^+ \\ m^- = n^- + 2 \end{cases} \quad \text{or} \quad \begin{cases} m^+ = n^+ + 2 \\ m^- = n^- \end{cases}$$

respectively. First assume the first case. Then

$$q(G)+q(H)+q(M)+q(M') = 2m+1+\frac{2m^+(m^++1)+m^-(m^-+1)+(m^--2)(m^--1)}{2} \equiv$$

$$\equiv m^+(m^++1)+\frac{2(m^-)^2-2m^-+2}{2}+1 \equiv$$

$$\equiv m^+(m^++1)+m^-(m^--1) \equiv 0 \mod 2.$$

Observing symmetry we get the same result for the second case. We conclude that

$$(-1)^{q(G)+q(H)+q(M)+q(M')} = 1,\ A = \emptyset \ \text{ or } \ \{1,2\}\,.$$

We now compute the sign between Δ_{j_M,B_M} and the Whittaker normalization. Applying the $q = 0$ case of Corollary 4.5.14, this sign is

$$(-1)^{\lceil n^+/2 \rceil}.$$

Also applying the $q = 2$ case of Corollary 4.5.14, we see that the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lceil m^+/2 \rceil + 1}.$$

Thus

$$♫(A) = (-1)^{q(G)+q(H)+q(M)+q(M')+\lceil n^+/2 \rceil + \lceil m^+/2 \rceil + 1}.$$

We have

$$♫(\emptyset) = (-1)^{\lceil n^+/2\rceil + \lceil m^+/2\rceil + 1} = (-1)^{\lceil m^+/2\rceil + \lceil m^+/2\rceil + 1} = -1.$$

$$♫(\{1,2\}) = (-1)^{\lceil n^+/2\rceil + \lceil m^+/2\rceil + 1} = (-1)^{\lceil (m^+-2)/2\rceil + \lceil m^+/2\rceil + 1} = 1.$$

This finishes the proof in this case.

Even case M_1:

Now $q(G), q(H)$ are even, and $q(M), q(M')$ are odd.

When $A = \emptyset$ or $\{1,2\}$, we have

$$\begin{cases} m^+ = n^+ \\ m^- = n^- + 2 \end{cases} \quad \text{or} \quad \begin{cases} m^+ = n^+ + 2 \\ m^- = n^- \end{cases}$$

respectively.

Similarly as in the even case M_{12} treated before, the sign between Δ_{j_M, B_M} and the Whittaker normalization is

$$(-1)^{\lfloor n^-/2\rfloor},$$

and the sign between $\Delta_{j, B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lfloor m^-/2\rfloor}.$$

Thus

$$♫(A) = (-1)^{\lfloor n^-/2\rfloor + \lfloor m^-/2\rfloor}.$$

We have

$$♫(\emptyset) = (-1)^{\lfloor (m^- - 2)/2\rfloor + \lfloor m^-/2\rfloor} = -1.$$

$$♫(\{1,2\}) = (-1)^{\lfloor m^-/2\rfloor + \lfloor m^-/2\rfloor} = 1.$$

This finishes the proof in this case.

Odd case M_2:

Now

$$q(G) = \frac{(2m-1)2}{2} = 2m-1,$$

$$q(H) = \frac{m^+(m^++1)+m^-(m^-+1)}{2},$$

$$q(M) = \frac{d-3}{2} = m-1,$$

$$q(M') = \frac{n^+(n^++1)+n^-(n^-+1)}{2}.$$

When $A = \emptyset$ or $\{1\}$, we have

$$\begin{cases} m^+ = n^+ \\ m^- = n^- + 1 \end{cases} \quad \text{or} \quad \begin{cases} m^+ = n^+ + 1 \\ m^- = n^- \end{cases}$$

respectively. Assume $A = \emptyset$, then

$$q(G)+q(H)+q(M)+q(M') = 3m-2+\frac{2m^+(m^++1)+m^-(m^-+1)+(m^--1)m^-}{2} \equiv$$

$$\equiv m + m^+(m^++1) + \frac{2(m^-)^2}{2} \equiv$$

$$\equiv m + m^+(m^++1) + (m^-)^2 \equiv m + m^- \equiv m^+ \mod 2.$$

Similarly, for $A = \{1\}$, we have

$$q(G)+q(H)+q(M)+q(M') \equiv m^- \mod 2.$$

We now compute the sign between Δ_{j_M,B_M} and the Whittaker normalization. Applying the $q = 1$ case of Corollary 4.5.14, this sign is

$$(-1)^{\lfloor n^+/2 \rfloor}.$$

Also applying the $q = 2$ case of Corollary 4.5.14, we see that the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lceil m^+/2 \rceil + 1}.$$

Thus

$$♫(A) = (-1)^{q(G)+q(H)+q(M)+q(M')+\lfloor n^+/2 \rfloor + \lceil m^+/2 \rceil + 1}.$$

We have

$$♫(\emptyset) = (-1)^{m^+ + \lfloor n^+/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^+ + \lfloor m^+/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^+ + m^+ + 1} = -1.$$

$$♫(\{1\}) = (-1)^{m^- + \lfloor n^+/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^- + \lfloor (m^+ - 1)/2 \rfloor + \lceil m^+/2 \rceil + 1} = (-1)^{m^-}.$$

This finishes the proof of in this case.

$\square$

Definition 6.9.4. Define

$$☼(A) := \begin{cases} 1, & A = \emptyset \text{ or } \{2\} \\ -1, & A = \{1\} \text{ or } \{1,2\} \end{cases}$$

Definition 6.9.5. Let $\mathfrak{e}, \gamma'$ be as in Definition 6.7.6. Suppose for each A as in Lemma 6.6.5, we have an element $g(A) \in \mathcal{H}^{\mathrm{ur}}(M'(\mathbb{Q}_p))$. Define

$$J(\mathfrak{e}, \gamma', A \mapsto g(A)) := \sum_A \text{☼}(A) SO_{\gamma'}(g(A)) \epsilon_R(j_M(\gamma'^{-1})) \epsilon_{R_H}(\gamma'^{-1}) \Phi_M^G(j_M(\gamma')^{-1}, \Theta^H_{\varphi_{\mathbb{V}^*}}),$$

where $\sum_A$ is the same as in Lemma 6.6.5.

Definition 6.9.6. Let $\mathfrak{e}, \gamma', \gamma_M$ be as in Definition 6.7.6. Define

$$Q(\mathfrak{e}, \gamma') = \bar{\iota}^{M'}(\gamma')^{-1} (\Delta^M_{M'})^{p,\infty}_{\emptyset}(\gamma', \gamma_M) O^{s'_M}_{\gamma_M}(f^{p,\infty}_M) \tau(M) k(M) k(G)^{-1} (-1)^{\dim A_{M'}} \bar{v}({M'}^0_{\gamma'})^{-1} \times$$

$$\times (-1)^{q(G_{\mathbb{R}})} \Delta^{\emptyset}_{j_M, B_M}(\gamma', j_M(\gamma')).$$

Here, as in Definition 6.7.6, the choice of $\gamma_M \in M(\mathbb{A}_f^p)$, of which γ' is an image, does not affect the definition.

Corollary 6.9.7. *With J defined in Definition 6.9.5 and Q defined in 6.9.6, we have*

$$I(\mathfrak{e}, \gamma') = Q(\mathfrak{e}, \gamma') J(\mathfrak{e}, \gamma', A \mapsto f^H_{p,M'}).$$

Proof. By (6.8.2) and Proposition 6.9.3, we have $\text{☼}(A) = \text{♫}(A)\,\text{♫}(\emptyset)^{-1} \det(\omega_{0,*})$. The corollary follows from Lemma 6.7.9, Corollary 6.8.7, Remark 6.9.2. $\square$

6.10 Symmetry of order n^G_M

Definition 6.10.1. Define a subgroup $\mathfrak{W} \subset \mathrm{Aut}((M')^{\mathrm{GL}}) = \mathrm{Aut}(M^{\mathrm{GL}})$ as follows: When $M = M_{12}$, so that $M^{\mathrm{GL}} = (M')^{\mathrm{GL}} = \mathbb{G}_m^2$, we define

$$\mathfrak{W} := \{\pm 1\}^2 \rtimes \mathfrak{S}_2,$$

where each factor $\{\pm 1\}$ acts on each $\mathbb{G}_m$ non-trivially and $\mathfrak{S}_2$ acts by swapping the two factors of $\mathbb{G}_m$. When $M = M_1$, so that $M^{\mathrm{GL}} = (M')^{\mathrm{GL}} = \mathrm{GL}_2$, we define

$$\mathfrak{W} := \mathbb{Z}/2\mathbb{Z}$$

where the non-trivial element acts on GL_2 by transpose inverse. When $M = M_2$ in the odd case, so that $M^{\mathrm{GL}} = (M')^{\mathrm{GL}}$, we define

$$\mathfrak{W} := \mathbb{Z}/2\mathbb{Z}.$$

When the context is clear we also view $\mathfrak{W}$ as a subgroup of $\mathrm{Aut}(M)$ or $\mathrm{Aut}(M')$, by extending its action trivially to M^{SO} or $(M')^{\mathrm{SO}}$.

Remark 6.10.2. As a subgroup of $\mathrm{Aut}(M)$, $\mathfrak{W}$ consists of automorphisms defined over $\mathbb{Q}$ and is contained in the image of the natural map

$$(\mathrm{Nor}_G\, M)(\mathbb{Q}) \to \mathrm{Aut}(M).$$

Lemma 6.10.3. $|\mathfrak{W}| = n_M^G$.

Proof. This follows from Lemma 6.6.4. $\square$

6.10.4

The group $\mathfrak{W}$ acts on the set of stable conjugacy classes in $M'(\mathbb{Q})$ preserving the following conditions

- being $\mathbb{R}$-elliptic.
- being (M, M')-regular.

- being an image of a semi-simple element in $M(\mathbb{A}_f^p)$.

Therefore we may and will assume that the sets $\Sigma(M')$ and $\Sigma(M')_1$ chosen in Definition 6.7.1 and Definition 6.7.5 are stable under $\mathfrak{W}$.

Lemma 6.10.5. *Consider the term $Q(\mathfrak{e}, \gamma')$ in Definition 6.9.6. It is invariant under the action of $\mathfrak{W}$ on the variable γ'.*

Proof. By (6.7.2), we have

$$Q(\mathfrak{e}, \gamma') = C \times SO_{\gamma'}(f_{M'}^{H(\emptyset),p,\infty})\Delta^{\emptyset}_{j_M,B_M}(\gamma', j_M(\gamma')),$$

where C is an expression that is obviously invariant $\mathfrak{W}$, and $H(\emptyset)$ is the endoscopic datum for G induced by the G-endoscopic datum for M that is parametrized by $\mathfrak{e} \in \mathscr{E}(M)$ and $A = \emptyset$. Note that the subgroup $\mathfrak{W} \subset \mathrm{Aut}(M')$ is contained in the image of[23]

$$(\mathrm{Nor}_{H(\emptyset)} M')(\mathbb{Q}) \to \mathrm{Aut}(M').$$

We first check that $\gamma' \mapsto SO_{\gamma'}(f_{M'}^{H(\emptyset),p,\infty})$ is invariant under $\mathfrak{W}$. Denote $\phi := f^{H(\emptyset),p,\infty}$. Let $w \in \mathfrak{W}$ be an arbitrary element. It suffices to check that $\phi_{M'}$ and $(\phi_{M'})^w$ have the same orbital integrals, where $(\phi_{M'})^w(g) := \phi_{M'}(w(g))$. By the density of regular semi-simple orbital integrals, it suffices to check that for all $(H(\emptyset), M')$-regular elements $g \in M'(\mathbb{A}_f^p)$, we have

$$O_g(\phi_{M'}) = O_{w(g)}(\phi_{M'}).$$

Without loss of generality we may replace $\mathbb{A}_f^p$ by a local field $F = \mathbb{Q}_v, v \neq p, \infty$. By [71, Lemma 6.1] (which follows [75]), we know that

$$O_g(\phi_{M'}) = D_{M'}^{H(\emptyset)}(g)^{1/2} O_g(\phi)$$

[23]This is not true if $H(\emptyset)$ is replaced by $H(\{1\})$ for instance when $M = M_{12}$.

and

$$O_{w(g)}(\phi_{M'}) = D_{M'}^{H(\emptyset)}(w(g))^{1/2} O_{w(g)}(\phi).$$

Here $D_{M'}^{H(\emptyset)}(g) := \prod_\alpha |1-\alpha(g)|_{\mathbb{A}_f^p}$, where the product is over the roots α of $H(\emptyset)_{\bar{F}}$ with respect to an arbitrary maximal torus of $H(\emptyset)_{\bar{F}}$ containing g, such that α is not a root of $M'_{\bar{F}}$ and $\alpha(g) \neq 1$.

Now because w is induced by an element in $(\mathrm{Nor}_{H(\emptyset)} M')(\mathbb{Q})$, we know that $D_{M'}^{H(\emptyset)}(g) = D_{M'}^{H(\emptyset)}(w(g))$ and $O_g(\phi) = O_{w(g)}(\phi)$. Thus $O_g(\phi_{M'}) = O_{w(g)}(\phi_{M'})$ as desired, and we have proved that $\gamma' \mapsto SO_{\gamma'}(f_{M'}^{H(\emptyset),p,\infty})$ is invariant under w.

We are left to check that for all $w \in \mathfrak{W}$, we have

$$\Delta^{\emptyset}_{j_M,B_M}(\gamma', j_M(\gamma')) = \Delta^{\emptyset}_{j_M,B_M}(w(\gamma'), j_M(w(\gamma'))),$$

or equivalently

$$\Delta_{j_M,B_M}(\gamma', j_M(\gamma')) = \Delta_{j_M,B_M}(w(\gamma'), j_M(w(\gamma'))) \tag{6.10.1}$$

By the assumption on γ' in Definition 6.7.1 and the assumption on $T_{M'}$ in the beginning of §6.8, we know that both γ' and $w(\gamma')$ lie in $T_{M'}$. Also both γ' and $w\gamma'$ are (M, M')-regular. Recall the formula for Δ_{j_M,B_M} from [40, p. 184], which is valid for any (M, M')-regular element $t \in T_{M'}(\mathbb{R})$:

$$\Delta_{j_M,B_M}(t, j_M(t)) = (-1)^{q(M'_{\mathbb{R}})+q(M_{\mathbb{R}})} \chi_{M,M'}(j_M(t)) \prod_\alpha (1-\alpha(j_M(t))),$$

where α runs through the B_M-positive roots of $T_{M,\mathbb{C}}$ in $M_{\mathbb{C}}$ that do not come from M' via j_M. The quasi-character $\chi_{M,M'}$ on $T_M(\mathbb{R})$, whose construction is given in loc. cit. (with the notation $\chi_{G,H}$), obviously restricts to the trivial character on $T_{M^{\mathrm{GL}}}$. Thus we only need to check that the term

$$T(t) := \prod_{\alpha}(1 - \alpha(j_M(t)))$$

is the same for $t = \gamma'$ and $t = w(\gamma')$. Since M is a direct product $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$, and since all the roots of M^{GL}, if any, come from M', we see that the value of $T(t)$ does not depend on the component of t in $(M')^{\mathrm{GL}}$. This finishes the proof.

□

Corollary 6.10.6. *For each $\mathfrak{e} = (M', s'_M, \eta_M) \in (M)$, choose a set $\Sigma(M')_1$ as in Definition 6.7.5 and §6.10.4. Let $\Sigma(M')_2$ be an arbitrary set of representatives of the $\mathfrak{W}$-orbits in $\Sigma(M')_1$. We have*

$$n_M^G \operatorname{Tr}'_M = \sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'\in\Sigma(M')_2} Q(\mathfrak{e},\gamma') \sum_{w\in\mathfrak{W}} J(\mathfrak{e}, w(\gamma'), A \mapsto f^H_{p,M'}).$$

Proof. This is a consequence of Lemma 6.7.7, Corollary 6.9.7, Lemma 6.10.5. □

6.11 Computation of J

We compute $J(\mathfrak{e}, \gamma', A \mapsto f^H_{p,M'})$ in Definition 6.9.5 using results from 5.4.

Lemma 6.11.1. *When $M = M_{12}$ in the odd case, we have*

$$\epsilon_R(j_M(\gamma'^{-1})) = \epsilon_{R_H}(\gamma'^{-1}),\ \ A = \{1,2\}\ \ \textit{or}\ \emptyset$$

$$\epsilon_R(j_M(\gamma'^{-1}))|_{A=\{1\}} = \epsilon_R(j_M(\gamma'^{-1}))|_{A=\{2\}}$$

$$\epsilon_{R_H}(\gamma'^{-1})|_{A=\{1\}} = \epsilon_{R_H}(\gamma'^{-1})|_{A=\{2\}}.$$

In all the other cases, we have

$$\epsilon_R(j_M(\gamma'^{-1})) = \epsilon_{R_H}(\gamma'^{-1}).$$

Proof. Easy consequences of the definition. □

Definition 6.11.2. Write $p^* := p^{a(d-2)/2}$. Write

$$f^H_{p,M'} = p^* k(A) + p^* h,$$

as in Proposition 5.4.3. When $M = M_{12}$, further write

$$k(A) = k_1(A) + k_2(A),$$

where $k_i(A)$ has Satake transform $\epsilon_i(A)(X_i^a + X_i^{-a})$, as in Proposition 5.4.3. Thus we have

$$J(\mathfrak{e}, \gamma', A \mapsto f^H_{p,M'}) = p^* J(\mathfrak{e}, \gamma', k) + p^*(\mathfrak{e}, \gamma', h),$$

and when $M = M_{12}$ we further have

$$J(\mathfrak{e}, \gamma', A \mapsto f^H_{p,M'}) = p^* J(\mathfrak{e}, \gamma', k_1) + p^* J(\mathfrak{e}, \gamma', k_2) + p^* J(\mathfrak{e}, \gamma', h).$$

Here we have abbreviated

$$J(\mathfrak{e}, \gamma', k) := J(\mathfrak{e}, \gamma', A \mapsto k(A)),$$

etc.

6.11.3 Some abbreviations

Write

$$\Phi^G_M := \Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}).$$

$$\Phi^G_{M,\text{endosc}} := \Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*})_{\text{endosc}}.$$

$$\epsilon_R\epsilon_{R_H} := \epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1}).$$

6.11.4 Odd case M_{12}

With the above abbreviations, it follows from Lemma 6.11.1 that we have

$$J(\mathfrak{e}, \gamma', h) = SO_{\gamma'}(h) \sum_A \text{☼}(A)\epsilon_R\epsilon_{R_H}\Phi_M^G(\Theta^H_{\varphi_{\mathbb{V}^*}}) =$$

$$= SO_{\gamma'}(h)\left[\Phi_M^G - \Phi_M^G - (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{endosc}} + (\epsilon_R\epsilon_{R_H})|_{A=\{2\}}\Phi^G_{M,\mathrm{endosc}}\right] = 0.$$

Similarly

$$J(\mathfrak{e}, \gamma', k_1) = SO_{\gamma'}(k_1(\emptyset))\left[\Phi_M^G + \Phi_M^G + (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{endosc}} + (\epsilon_R\epsilon_{R_H})|_{A=\{2\}}\Phi^G_{M,\mathrm{endosc}}\right] =$$

$$= 2SO_{\gamma'}(k_1(\emptyset))\left[\Phi_M^G + (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{endosc}}\right]$$

and

$$J(\mathfrak{e}, \gamma', k_2) = 2SO_{\gamma'}(k_2(\emptyset))\left[\Phi_M^G - (\epsilon_R\epsilon_{R_H})|_{A=\{1\}}\Phi^G_{M,\mathrm{endosc}}\right]$$

6.11.5 Even case M_{12}

With similar computations as above, we get

$$J(\mathfrak{e}, \gamma', h) = 0$$

$$J(\mathfrak{e}, \gamma', k_1) = 2SO_{\gamma'}(k_1(\emptyset))\Phi_M^G$$

$$J(\mathfrak{e}, \gamma', k_2) = 2SO_{\gamma'}(k_2(\emptyset))\Phi_M^G.$$

6.11.6 Case M_1 and odd case M_2

Similar computations give

$$J(\mathfrak{e},\gamma',h)=0$$

$$J(\mathfrak{e},\gamma',k)=2SO_{\gamma'}(k(\emptyset))\Phi_M^G.$$

6.12 Breaking symmetry, case M_{12}

We keep the notation in Corollary 6.10.6 and §6.11.

Definition 6.12.1. We say that an element in $M'(\mathbb{R})$ is *good at* ∞ its component in $(M')^{\mathrm{GL}}(\mathbb{R})=\mathbb{R}^\times\times\mathbb{R}^\times$ has two equal signs. We say that an element in $M'(\mathbb{Q}_p)$ is *good at* p if its component in $(M')^{\mathrm{GL}}(\mathbb{Q}_p)=\mathbb{Q}_p^\times\times\mathbb{Q}_p^\times$ has p-adic valuations $(-a,0)$. Here the first $\mathbb{G}_m$-factor is $\mathrm{GL}(V_1)$ and the second is $\mathrm{GL}(V_2/V_1)$.

Corollary 6.12.2. *Let $M=M_{12}$. We have*

$$n_M^G\,\mathrm{Tr}'_M=\sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)^c}|\mathrm{Out}_M(\mathfrak{e})|^{-1}\sum_{\gamma'}Q(\mathfrak{e},\gamma')4p^*J(\mathfrak{e},\gamma',k_1),$$

where γ' runs through the elements in $\Sigma(M')_1$ that are good at ∞ and good at p.

Proof. We start with the formula for the LHS in Corollary 6.10.6. Fix a pair $(\mathfrak{e},\gamma')$ in that formula.

We first treat the odd case. Let $w_1:=(-1,1)\in\{\pm1\}\times\{\pm1\}\subset\mathfrak{W}=\{\pm1\}^2\rtimes\mathfrak{S}_2$. Let w_{12} be the non-trivial element in $\mathfrak{S}_2\subset\mathfrak{W}$. Namely, w_1 inverts the first coordinate of $\mathbb{G}_m^2$ and w_{12} swaps the two coordinates. Using the computations of $J(\mathfrak{e},\gamma',k_1)$ and $J(\mathfrak{e},\gamma',k_2)$ in §6.11, it follows from the vanishing statement in Proposition 4.11.8 that

$$J(\mathfrak{e},\gamma',k_1)=J(\mathfrak{e},\gamma',k_1)=0$$

unless γ' is good at ∞. Suppose γ' is good at ∞. Then by Proposition 4.11.9, Proposition 4.11.10, and the computation in §6.11, we have

$$J(\mathfrak{e},\gamma',k_1)=J(\mathfrak{e},w_{12}(\gamma'),k_2) \tag{6.12.1}$$

$$J(\mathfrak{e},\gamma',k_1)=J(\mathfrak{e},w_1(\gamma'),k_1) \tag{6.12.2}$$

because the functions $k_1(\emptyset)$ and $k_2(\emptyset)$ are pull-backs of each other under w_{12}, and the function $k_1(\emptyset)$ is invariant under w_1. From §6.11 we have

$$J(\mathfrak{e},\gamma',A\mapsto f^H_{p,M'})=p^*J(\mathfrak{e},\gamma',k_1)+p^*J(\mathfrak{e},\gamma',k_2).$$

Therefore, if

$$\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),A\mapsto f^H_{p,M'})\neq 0,$$

then γ' is good at ∞, and moreover from (6.12.1) we know that there exists $w\in\mathfrak{W}$ such that

$$J(\mathfrak{e},w(\gamma'),k_1)\neq 0.$$

By §6.11, the last condition implies that $SO_{w(\gamma')}(k_1(\emptyset))\neq 0$, and it follows easily that either $w(\gamma')$ or $w_1(w(\gamma'))$ is good at p. Note that in the orbit $\mathfrak{W}\gamma'$ there is at most one element that is good at p. Without loss of generality assume γ' is that element. Then we have

$$(p^*)^{-1}\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),A\mapsto f^H_{p,M'})=\sum_{w\in\mathfrak{W}} J(\mathfrak{e},w(\gamma'),k_1)+J(\mathfrak{e},w(\gamma'),k_2)=$$

$$\overset{(6.12.1)}{=\!=\!=}\sum_{w\in\mathfrak{W}} 2J(\mathfrak{e},w(\gamma'),k_1)=2J(\mathfrak{e},\gamma',k_1)+2J(\mathfrak{e},w_1(\gamma'),k_1)\overset{(6.12.2)}{=\!=\!=}4J(\mathfrak{e},\gamma',k_1).$$

The corollary in the odd case follows.

The even case is treated with almost identical arguments. The only differences are in that we now use the vanishing statement in Proposition 4.11.15 rather than Proposition 4.11.8, and in that we simply use the invariance of $\Phi_M^G(\cdot, \Theta_{\mathbb{V}^*})$ under $(\mathrm{Nor}_G M)(\mathbb{R})$ to deduce (6.12.1) and (6.12.2) . $\square$

6.13 Breaking symmetry, case M_1 and odd case M_2

We keep the notation in Corollary 6.10.6 and §6.11.

Definition 6.13.1. Suppose $M = M_1$. We say that an element in $M'(\mathbb{Q}_p)$ is *good at* p if its component in $(M')^{\mathrm{GL}}(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)$ has determinant of p-adic valuations $-a$. We say that any element in $M'(\mathbb{R})$ is *good at* ∞.

Suppose $M = M_2$ in the odd case. We say that an element in $M'(\mathbb{Q}_p)$ is *good at* p if its component in $(M')^{\mathrm{GL}}(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ has valuation $-a$. We say that an element in $M'(\mathbb{R})$ is *good at* ∞ if its component in $(M')^{\mathrm{GL}}(\mathbb{R}) = \mathbb{R}^\times$ is positive.

Corollary 6.13.2. *Suppose $M = M_1$, or $M = M_2$ in the odd case. We have*

$$n_M^G \operatorname{Tr}'_M = \sum_{\mathfrak{e}=(M', s'_M, \eta_M) \in \mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'} Q(\mathfrak{e}, \gamma') 2p^* J(\mathfrak{e}, \gamma', k),$$

where γ' runs through the elements in $\Sigma(M')_1$ that are good ∞ and good at p.

Proof. We start with the formula for the LHS in Corollary 6.10.6. Fix a pair $(\mathfrak{e}, \gamma')$ in that formula. Let $w_0 \in \mathfrak{W}$ be the non-trivial element. Using the computation of $J(\mathfrak{e}, \gamma', k)$ in §6.11, we see that if it is non-zero, then $\Phi_M^G(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}) \neq 0$, and by the vanishing statement in Proposition 4.10.1 this implies that γ' is good at ∞ when $M = M_2$. Moreover, it follows from the invariance of $k(\emptyset)$ under w_0 and the invariance of $\Phi_M^G(\cdot, \Theta_{\mathbb{V}^*})$ under

$(\mathrm{Nor}_G\, M)(\mathbb{R})$ that we have

$$J(\mathfrak{e}, \gamma', k) = J(\mathfrak{e}, w_0(\gamma'), k). \tag{6.13.1}$$

From §6.11 we have

$$J(\mathfrak{e}, \gamma', A \mapsto f^H_{p,M'}) = p^* J(\mathfrak{e}, \gamma', k).$$

Therefore by (6.13.1) we have

$$\sum_{w \in \mathfrak{W}} J(\mathfrak{e}, w(\gamma'), A \mapsto f^H_{p,M'}) = 2p^* J(\mathfrak{e}, \gamma', k).$$

If this is non-zero, then $SO_{\gamma'}(k(\emptyset)) \neq 0$, and it follows easily that either γ' or $w_0\gamma'$ is good at p. Note that at most one of them can be good at p. The corollary follows. $\square$

6.14 Main computation

Proposition 6.14.1. *Let M be a standard Levi subgroup of G. When $a \in \mathbb{Z}_{>0}$ is large enough (for a fixed $f^{p,\infty}$), we have*

$$\begin{aligned} \mathrm{Tr}'_M = 4p^* \sum_{\mathfrak{e}=(M', s'_M, \eta_M) \in \mathscr{E}(M)^c} & (-1)^{\dim A_{M'} + q(G_{\mathbb{R}})} \left|\mathrm{Out}_M(\mathfrak{e})\right|^{-1} \\ & \sum_{\gamma'} Q(\mathfrak{e}, \gamma') SO_{\gamma'}(k_1(\emptyset)) L_M(j_M(\gamma')), \end{aligned} \tag{6.14.1}$$

where γ' runs through the elements in $\Sigma(M')_1$ that are good at ∞ and good at p. Here we understand that $k_1(\emptyset) := k(\emptyset)$ when $M = M_1$ or M_2.

Proof. Note that $(-1)^{\dim A_{M'}}$ only depends on M. In fact,

$$(-1)^{\dim A_{M'}} = \begin{cases} 1, & M = M_{12} \\ -1, & \text{otherwise} \end{cases}.$$

This dichotomy is to be compared with the dichotomy of signs in Propositions 4.11.8, 4.11.15 for M_{12}, against Propositions 4.9.1, 4.10.1 for M_1 and M_2. We first treat the odd case M_{12}. By Lemma 6.6.4, Corollary 6.12.2, and §6.11, we have

$$8\,\mathrm{Tr}'_M = \sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'} Q(\mathfrak{e},\gamma') 4p^* J(\mathfrak{e},\gamma',k_1) =$$

$$= \sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'} Q(\mathfrak{e},\gamma') 8p^* SO_{\gamma'}(k_1(\emptyset)) \times$$

$$\times \left[\Phi^G_M(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}) + \epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})|_{A=\{1\}} \Phi^G_M(j_M(\gamma'^{-1}), \Theta_{\mathbb{V}^*})_{\mathrm{endosc}} \right],$$

where γ' runs through the elements in $\Sigma(M')_1$ that are good at ∞ and good at p. Suppose that γ' contributes non-trivially to the above sum. Then $Q(\mathfrak{e},\gamma') \neq 0$. From Definition 6.9.6, we have

$$O^{s'_M}_{\gamma_M}(f^{p,\infty}_M) \neq 0.$$

Therefore the component of γ_M in $M^{\mathrm{GL}}(\mathbb{A}^p_f) = (M')^{\mathrm{GL}}(\mathbb{A}^p_f)$ lies in a compact subset that only depends on $f^{p,\infty}$ and not on a. Note that because γ' is an image of γ_M, the component of γ' in $(M')^{\mathrm{GL}}(\mathbb{Q})$ is equal to the component of γ_M in $M^{\mathrm{GL}}(\mathbb{A}^p_f)$. When a is large enough, we deduce from this observation and the assumption that γ' is good at p that the real absolute value of the component of γ' in the first $\mathbb{G}_m$ is strictly smaller than the ± 1-st power of that of the second. In other words, $j_M(\gamma')$ is in the range $x_1 < -|x_2|$ considered in Proposition

4.11.8 and Proposition 4.11.15. Observe

$$\Phi_M^G(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*}) = \Phi_M^G(j_M(\gamma'), \Theta_{\mathbb{V}})$$

$$\Phi_M^G(j_M(\gamma')^{-1}, \Theta_{\mathbb{V}^*})_{\mathrm{endosc}} = \Phi_M^G(j_M(\gamma'), \Theta_{\mathbb{V}})_{\mathrm{endosc}}$$

and

$$\epsilon_R(j_M(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1}) = 1 \quad \text{for } j_M(\gamma'^{-1}) \text{ in the range mentioned above.}$$

The proposition now follows from Proposition 4.11.8.

The even case M_{12}, odd and even case M_1, and odd case M_2, are proved in exactly the same way, by applying Lemma 6.6.4, the corresponding computation in §6.11, Corollary 6.13.2, Proposition 4.11.15, Proposition 4.9.1, and Proposition 4.10.1. When $M = M_1$, we only know that the component of γ' in $M^{\mathrm{GL}}(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$ is (stably) conjugate to the component of γ_M in $(M')^{\mathrm{GL}}(\mathbb{A}_f^p) = \mathrm{GL}_2(\mathbb{A}_f^p)$ (as opposed to that they are equal), but that already implies that they have equal determinant. Again from the assumption that a is large and γ' is good at p, we deduce that $j_M(\gamma')$ is in the range $a^2 + b^2 < 1$ considered in Proposition 4.9.1.[24] When $M = M_2$ in the odd case, we deduce that $j_M(\gamma')$ is in the range $0 < a < 1$ considered in Proposition 4.10.1 in the same way as when $M = M_{12}$. □

We now plug the definition of $Q(\mathfrak{e}, \gamma')$ in Definition 6.9.6 into the formula (6.14.1):

(6.14.2)

$$\mathrm{Tr}'_M = 4p^*\tau(M)k(M)k(G)^{-1} \sum_{\mathfrak{e}=(M', s'_M, \eta_M)\in\mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'} SO_{\gamma'}(k_1(\emptyset))L_M(j_M(\gamma'))$$
$$\bar{\iota}^{M'}(\gamma')^{-1}(\Delta_{M'}^M)_\emptyset^{p,\infty}(\gamma', \gamma_M)O_{\gamma_M}^{s'_M}(f_M^{p,\infty})\bar{v}(M'^0_{\gamma'})^{-1}\Delta^\emptyset_{j_M, B_M}(\gamma', j_M(\gamma')).$$

[24]Note that the two symbols a in this sentence have different meanings.

Observe that when γ' is good at p, we have:

$$SO_{\gamma'}(k_1(\emptyset)) = SO_{\gamma'}(k_a \otimes 1_{(M')^{\mathrm{SO}}}), \tag{6.14.3}$$

where (using the notation of Proposition 5.4.3)

(6.14.4)

$$\mathcal{H}^{\mathrm{ur}}((M')^{\mathrm{GL}}(\mathbb{Q}_p)) \ni k_a := \begin{cases} -X_1^{-a}, \ M = M_{12} \text{ or } M_2 \text{ (so that } (M')^{\mathrm{GL}} = \mathbb{G}_m^2 \text{ or } \mathbb{G}_m \text{ resp.)} \\ -X_1^{-a} - X_2^{-a}, \ M = M_1 \text{ (so that } (M')^{\mathrm{GL}} = \mathrm{GL}_2) \end{cases}$$

and $1_{(M')^{\mathrm{SO}}}$ denotes the unit in $\mathcal{H}^{\mathrm{ur}}((M')^{\mathrm{SO}}(\mathbb{Q}_p))$. Let γ'_{GL} (resp. γ'_{SO}) be the component of γ' in $(M')^{\mathrm{GL}}$ (resp. $(M')^{\mathrm{SO}}$). Then we can rewrite (6.14.3) as

$$SO_{\gamma'}(k_1(\emptyset)) = SO_{\gamma'_{\mathrm{GL}}}(k_a) SO_{\gamma'_{\mathrm{SO}}}(1_{(M')^{\mathrm{SO}}}). \tag{6.14.5}$$

Consider the natural bijection

$$\mathscr{E}(M)^c \xrightarrow{\sim} \mathscr{E}(M^{\mathrm{SO}})^c$$

$$(M', s'_M = (1, s^{\mathrm{SO}}), \eta_M = 1 \times \eta^{\mathrm{SO}}) \mapsto ((M')^{\mathrm{SO}}, s^{\mathrm{SO}}, \eta^{\mathrm{SO}}).$$

Note that $\gamma'_{\mathrm{GL}}\gamma'_{\mathrm{SO}}$ is (M, M')-regular if and only if γ'_{SO} is $(M^{\mathrm{SO}}, (M')^{\mathrm{SO}})$-regular. We have assumed $\gamma' \in \Sigma(M')_1$, so this is the case. By the Fundamental Lemma (Proposition 6.1.1 (2)), we know that $SO_{\gamma'_{\mathrm{SO}}}(1) \neq 0$ only if γ'_{SO} is an image of a semi-simple element $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$, and in this case we have

$$SO_{\gamma'_{\mathrm{SO}}}(1_{(M')^{\mathrm{SO}}}) = (\Delta^{M^{\mathrm{SO}}}_{(M')^{\mathrm{SO}}})_p(\gamma'_{\mathrm{SO}}, \gamma_{p,\mathrm{SO}}) O^{s^{\mathrm{SO}}}(1_{M^{\mathrm{SO}}}), \tag{6.14.6}$$

where $(\Delta^{M^{\mathrm{SO}}}_{(M')^{\mathrm{SO}}})_p$ is the canonical unramified normalization of transfer factors and $1_{M^{\mathrm{SO}}}$ denotes the unit in $\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{SO}}(\mathbb{Q}_p))$.

Note that when γ'_{SO} is an image of $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$ as above, $\gamma' = \gamma'_{\mathrm{GL}}\gamma'_{\mathrm{SO}}$ is an image of $\gamma'_{\mathrm{GL}}\gamma_{p,\mathrm{SO}} \in M(\mathbb{Q}_p)$, and for the canonical unramified normalizations of transfer factors we have

$$(\Delta^{M^{\mathrm{SO}}}_{(M')^{\mathrm{SO}}})_p(\gamma'_{\mathrm{SO}}, \gamma_{p,\mathrm{SO}}) = (\Delta^M_{M'})_p(\gamma', \gamma'_{\mathrm{GL}}\gamma_{p,\mathrm{SO}}). \tag{6.14.7}$$

From (6.14.2) (6.14.5) (6.14.6) (6.14.7), we obtain

$$\begin{aligned}\mathrm{Tr}'_M = 4p^*\tau(M)k(M)k(G)^{-1} \sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \\ \sum_{\gamma'} \bar{\iota}^{M'}(\gamma')^{-1}\bar{v}(M'^0_{\gamma'})^{-1} SO_{\gamma'_{\mathrm{GL}}}(k_a) L_M(j_M(\gamma'))(\Delta^M_{M'})^{p,\infty}_{\emptyset}(\gamma', \gamma_M)\Delta^{\emptyset}_{j_M,B_M}(\gamma', j_M(\gamma')) \\ (\Delta^M_{M'})_p(\gamma', \gamma'_{\mathrm{GL}}\gamma_{p,\mathrm{SO}}) O^{s'_M}_{\gamma_M}(f^{p,\infty}_M) O^{s^{\mathrm{SO}}}_{\gamma_{p,\mathrm{SO}}}(1_{M^{\mathrm{SO}}}),\end{aligned} \tag{6.14.8}$$

where γ' runs through a set of representatives of the stable conjugacy classes in $M'(\mathbb{Q})$ that are $\mathbb{R}$-elliptic and good at ∞, such that γ' is an image from $G(\mathbb{A}_f)$. For each γ', we choose $\gamma_M \in M(\mathbb{Q}_p)$ and $\gamma_{p,\mathrm{SO}} \in M^{\mathrm{SO}}(\mathbb{Q}_p)$ such that γ' is an image of γ_M over $\mathbb{A}^p_f$ and an image of $\gamma'_{\mathrm{GL}}\gamma_{p,\mathrm{SO}}$ over $\mathbb{Q}_p$.

Lemma 6.14.2. *If $\gamma' \in M'(\mathbb{Q})$ is $\mathbb{R}$-elliptic and is an image of from $M(\mathbb{A}_f)$, then it is an image from $M(\mathbb{Q})$. Moreover, if $\gamma_\infty \in M(\mathbb{R})$ is a prescribe elliptic element of which γ' is an image, then γ' is an image of some $\gamma \in M(\mathbb{Q})$ such that γ is conjugate to γ_∞ in $M(\mathbb{R})$.*

Proof. We recall a construction from [43] in our setting. Let $\gamma^* \in M^*(\mathbb{Q})$ such that γ' is an image of it. Note that γ' is also an image from $M(\mathbb{R})$ since it is $\mathbb{R}$-elliptic. Let $\gamma_{\mathbb{A}} \in M(\mathbb{A})$ such that γ' is an image of it. When γ_∞ is prescribed as in the statement of the lemma, we take $\gamma_{\mathbb{A}}$ such that its archimedean component is γ_∞. From γ^* and $\gamma_{\mathbb{A}}$ Labesse ([43, §2.6], with $L = M, H = M^*$) constructs an element

$$\mathrm{obs} \in \mathfrak{E}(I^*, M^*; \mathbb{A}/\mathbb{Q}) := \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I^* \backslash M^*)/\, \mathbf{H}^0_{\mathrm{ab}}(\mathbb{A}, M^*),$$

generalizing the construction of Kottwitz in [38]. By [43, Théorème 2.6.3], it suffices to prove that we may modify $\gamma_{\mathbb{A}}$ by replacing $(\gamma_{\mathbb{A}})_v$ with another element stably conjugate to it over $\mathbb{Q}_v$, for some finite place v, to achieve $\mathrm{obs} = 1$. This follows from exactly the same argument as the second paragraph of [40, p. 188], noting that $\mathfrak{E}(I^*, M^*; \mathbb{Q})$ is isomorphic to the dual group of the finite abelian group $\mathfrak{K}(I^*/\mathbb{Q})$ (for $I^* \subset M^*$) from [38, §4.6], and that for any finite place v we have $\mathfrak{K}(I^*/\mathbb{Q}_v)^D \cong \mathfrak{E}(I^*, M^*; \mathbb{Q}_v) \cong \mathfrak{D}(I^*, M^*; \mathbb{Q}_v) = \ker(\mathbf{H}^1(\mathbb{Q}_v, I^*) \to \mathbf{H}^1(\mathbb{Q}_v, M^*))$ so that given any element of it we may use it to modify $(\gamma_{\mathbb{A}})_v$ to obtain an element stably conjugate to it over $\mathbb{Q}_v$. For more details also see [32]. $\square$

By the above lemma, we may assume that each γ' in (6.14.8) is an image of an element $\gamma \in M(\mathbb{Q})$, and that γ is conjugate to $j_M(\gamma')$ in $M(\mathbb{R})$. Note that we have $L_M(j_M(\gamma')) = L_M(\gamma)$. (In fact $L_M(\cdot)$ only depends on $\mathbb{C}$-conjugacy classes). Of course we may assume that the M^{GL}-component of γ is equal to γ'_{GL}. Thus we may take γ_M and $\gamma'_{\mathrm{GL}}\gamma_{p,\mathrm{SO}}$ to be localizations of γ in $M(\mathbb{A}_f^p)$ and $M(\mathbb{Q}_p)$ respectively.

Recall from Definition 6.9.1 that we have assumed that the transfer factors

$$(\Delta^M_{M'})^{p,\infty}_{\emptyset}, \quad \Delta^M_{M',p}, \quad \Delta^{\emptyset}_{j_M, B_M}$$

satisfy the global product formula up to the constant ♫$(\emptyset)$. We have seen ♫$(\emptyset) = -1$ in Proposition 6.9.3. Therefore

$$\left[(\Delta^M_{M'})^{p,\infty}_{\emptyset} \Delta^M_{M',p} \Delta^{\emptyset}_{j_M, B_M}\right](\gamma', \gamma) = -1.$$

We summarize the above discussion in the following proposition.

Proposition 6.14.3. *Let M be a standard Levi subgroup of G. When $a \in \mathbb{Z}_{>0}$ is large enough (for a fixed $f^{p,\infty}$), we have*

$$\mathrm{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1} \sum_{\mathfrak{e}=(M',s'_M,\eta_M)\in\mathscr{E}(M)^c} |\mathrm{Out}_M(\mathfrak{e})|^{-1} \sum_{\gamma'} \mathscr{A}(\gamma') \tag{6.14.9}$$

where γ' runs through the stable conjugacy classes in $M'(\mathbb{Q})$ such that it is (M, M')-regular and is an image from $M(\mathbb{Q})$, and such that its $(M')^{\mathrm{GL}}$-component γ'_{GL} is good at ∞. For each γ', there is an element γ of $M(\mathbb{Q})$ of which γ' is an image, and we have

$$\begin{aligned}\mathscr{A}(\gamma') = \bar{\iota}^M(\gamma)^{-1}\bar{v}(M^0_\gamma)^{-1} SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O^{s'_M}_{\gamma}(f^{p,\infty}_M) O^{s^{\mathrm{SO}}}_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}}})\times \\ \times\Delta^{\emptyset}_{j_M,B_M}(\gamma', j_M(\gamma'))\Delta^{\emptyset}_{j_M,B_M}(\gamma',\gamma)^{-1}.\end{aligned} \tag{6.14.10}$$

□

Definition 6.14.4. For a connected reductive group I over $\mathbb{R}$ that is cuspidal, define $\mathscr{D}(I)$ to be the cardinality of $\ker(\mathbf{H}^1(\mathbb{R}, T) \to \mathbf{H}^1(\mathbb{R}, I))$, where T is any elliptic maximal torus of I. This is also equal to the cardinality of each discrete series L-packet, and equal to the cardinality of the complex Weyl group divided by the cardinality of the real Weyl group for T.

Lemma 6.14.5. *If $M = M_1$ or M_{12}, then $M_{\mathbb{R}}$ has no endoscopy, in the sense that $\mathscr{D}(M) = 1$. In particular, any $\mathbb{R}$-elliptic elements that are stably conjugate to each other are conjugate over $\mathbb{R}$. If $M = M_2$ in the odd case, then $\mathscr{D}(M) = 2$. In particular, any $\mathbb{R}$-elliptic element $\gamma \in M(\mathbb{R})$ satisfies $\mathscr{D}(M^0_\gamma)\left|\mathfrak{D}(M^0_\gamma, M; \mathbb{R})\right| = 2$.*

Proof. Suppose $M = M_1$ or M_{12}. Then $M_{\mathbb{R}}$ is a product of GL_2 or $\mathbb{G}_m$ and a compact group, so $\mathfrak{D}(M) = 1$. For the second statement it suffices to prove that $\mathfrak{D}(I, M; \mathbb{R})$ is trivial for any reductive subgroup I of $M_{\mathbb{R}}$ that contains an elliptic maximal torus T of $M_{\mathbb{R}}$. By [38, Lemma 10.2] we know that $\mathbf{H}^1(\mathbb{R}, T)$ surjects onto both $\mathbf{H}^1(\mathbb{R}, I)$ and $\mathbf{H}^1(\mathbb{R}, G)$, so $\mathfrak{D}(T, G; \mathbb{R})$ surjects onto $\mathfrak{D}(I, G; \mathbb{R})$, implying that $\mathfrak{D}(I, M; \mathbb{R})$ is trivial. Now suppose $M = M_2$ in the odd case. We have $M_{\mathbb{R}} = \mathbb{G}_m \times \mathrm{SO}(n-1, 1)$, so $\mathscr{D}(M) = \mathscr{D}(\mathrm{SO}(n-1, 1)) = 2$.[25] Let $\gamma \in M(\mathbb{R})$ be an elliptic semi-simple element. Then $I := M^0_\gamma$ contains an elliptic maximal torus T of $M_{\mathbb{R}}$. As in the first part of the proof $\mathfrak{D}(T, M; \mathbb{R})$ surjects onto $\mathfrak{D}(I, M; \mathbb{R})$, so $\mathfrak{D}(I, M; \mathbb{R})$ has cardinality at most 2. If $|\mathfrak{D}(I, M; \mathbb{R})| = 1$, then $\mathscr{D}(I) = |\mathfrak{D}(T, I; \mathbb{R})| = |\mathfrak{D}(T, M; \mathbb{R})| = 2$. If $|\mathfrak{D}(I, M; \mathbb{R})| = 2$, then $\mathfrak{D}(T, I; \mathbb{R})$ cannot be non-trivial, because the non-trivial element of $\mathfrak{D}(I, M; \mathbb{R})$ has a preimage $x \in \mathfrak{D}(T, M; \mathbb{R})$ and we have $\{x\} \sqcup \mathfrak{D}(T, I; \mathbb{R}) \subset \mathfrak{D}(T, M; \mathbb{R})$. □

Corollary 6.14.6. *Let $\mathscr{C}_M = 1$ for $M = M_{12}$ or M_1, and let $\mathscr{C}_M = 2$ for $M = M_2$ (in the odd case). We have*

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^* \tau(M) k(M) k(G)^{-1} \sum_{\gamma_0} \sum_{\kappa} \bar{\iota}^M(\gamma_0)^{-1} \bar{v}(I_0)^{-1} SO_{\gamma_0, \mathrm{GL}}(k_a) L_M(\gamma_0) O^{\kappa}_{\gamma_0}(f^{p,\infty}_M) O^{\kappa^{\mathrm{SO}}}_{\gamma_0, \mathrm{SO}}(1_{M^{\mathrm{SO}}}) \sum_{x \in \mathfrak{D}(I_0, M; \mathbb{R})} \mathscr{D}(x) \langle \kappa, x \rangle, \tag{6.14.11}$$

where

- *γ_0 runs through a fixed set of representatives of the stable conjugacy classes in $M(\mathbb{Q})$ that are elliptic over $\mathbb{R}$ and good at ∞.*
- *$I_0 := M^0_{\gamma_0}$.*
- *κ runs through $\mathfrak{K}(I_0/\mathbb{Q}) = \mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})^D$.*
- *$\mathscr{D}(x) := \mathscr{D}(M^0_{x,\mathbb{R}})$.*

[25] To compute $\mathscr{D}(\mathrm{SO}(n-1,1))$, consider the elliptic maximal torus $T = \mathrm{SO}(2)^{(n-1)/2}$ of $\mathrm{SO}(n-1,1)$. It is inside the maximal compact subgroup $\mathrm{SO}(n-1)$ of $\mathrm{SO}(n-1,1)$, therefore ([45, §6.4]) $\mathscr{D}(\mathrm{SO}(n-1,1))$ is equal to the cardinality of the complex Weyl group of $\mathrm{SO}(n-1,1)$ divided by the cardinality of the complex Weyl group of $\mathrm{SO}(n-1)$, which is 2.

Proof. This is a consequence of the usual conversion from $(\mathfrak{e}, \gamma')$ to (γ_0, κ) (cf. [44, Corollary IV.3.6] and [32]) and the following arguments.

When $M = M_{12}$ or M_2, any set $\mathfrak{D}(I_0, M; \mathbb{R})$ in (6.14.11) is a singleton. Also for any γ' in (6.14.9), the formula (6.14.10) simplifies to

$$\mathscr{A}(\gamma') = \bar{\iota}^M(\gamma)^{-1}\bar{v}(M_\gamma^0)^{-1}SO_{\gamma_{\mathrm{GL}}}(k_a)L_M(\gamma)O_\gamma^{s'_M}(f_M^{p,\infty})O_{\gamma_{\mathrm{SO}}}^{s^{\mathrm{SO}}}(1_{M^{\mathrm{SO}}})$$

because γ is automatically conjugate to $j_M(\gamma')$ over $\mathbb{R}$.

Suppose $M = M_2$ in the odd case. Given γ_0 and κ as in (6.14.11), we first claim that the function

$$\langle \kappa, \cdot \rangle : \mathfrak{D}(I_0, M; \mathbb{R}) \to \mathbb{C}^\times$$

takes values in $\{\pm 1\}$. It suffices to prove the claim with $M_\mathbb{R} \cong \mathbb{G}_m \times \mathrm{SO}(n-1,1)$ replaced by $M_\mathbb{R}^{\mathrm{SO}} \cong \mathrm{SO}(n-1,1)$. Now the above function is the pull-back of a character on the abelian group

$$\mathfrak{E}(I_0^{\mathrm{SO}}, M^{\mathrm{SO}}; \mathbb{R}) := \ker(\mathbf{H}^1_{\mathrm{ab}}(\mathbb{R}, I_0^{\mathrm{SO}}) \to \mathbf{H}^1_{\mathrm{ab}}(\mathbb{R}, M^{\mathrm{SO}})),$$

where I_0^{SO} is the component of I^0 in M^{SO}. It suffices to show that $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{R}, I_0^{\mathrm{SO}})$ is 2-torsion. By [38, Lemma 10.2] and [8, Theorem 5.4] we know that $\mathbf{H}^1_{\mathrm{ab}}(\mathbb{R}, I_0^{\mathrm{SO}})$ is mapped onto by $\mathbf{H}^1(\mathbb{R}, T)$, where T is a torus in $I_{0,\mathbb{R}}^{\mathrm{SO}}$ that is elliptic maximal in $M_\mathbb{R}^{\mathrm{SO}}$. Then T is in fact a product of $\mathrm{U}(1)$ and $\mathbf{H}^1(\mathbb{R}, T)$ is 2-torsion. The claim is proved.

It follows from the above claim and Lemma 6.14.5 that we have

$$\sum_{x \in \mathfrak{D}(I_0, M; \mathbb{R})} \mathscr{D}(x)\langle \kappa, x\rangle = \begin{cases} 2, & \langle \kappa, \cdot \rangle \equiv 1 \text{ on } \mathfrak{D}(I_0, M; \mathbb{R}) \\ 0, & \text{otherwise} \end{cases}$$

Thus to prove the corollary it suffices to prove that for any γ' in (6.14.9), if it satisfies the following property † then $\mathscr{A}(\gamma') = 0$.

† For one (hence any) $\gamma \in M(\mathbb{Q})$ of which γ' is an image, the function $\langle s'_M, \cdot \rangle$ is non-trivial on $\mathfrak{D}(M^0_\gamma, M; \mathbb{R})$.

Take such a γ' satisfying condition †. Take $\gamma \in M(\mathbb{Q})$ such that γ' is an image of γ. Then we have (6.14.10), which is equivalent to

(6.14.12)

$$\mathscr{A}(\gamma') = \bar{\iota}^M(\gamma)^{-1}\bar{v}(M^0_\gamma)^{-1} SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O^{s'_M}_\gamma (f^{p,\infty}_M) O^{s^{\mathrm{SO}}}_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}}}) \langle s'_M, \mathrm{inv}(j_M(\gamma'), \gamma) \rangle.$$

By the condition † we know $\mathfrak{D}(M^0_\gamma, M; \mathbb{R})$ is non-trivial, and so by Lemma 6.14.5 it has cardinality 2 and for any elliptic maximal torus T of $M^0_{\gamma,\mathbb{R}}$ we have a bijection

$$\mathfrak{D}(T, M; \mathbb{R}) \xrightarrow{\sim} \mathfrak{D}(M^0_\gamma, M; \mathbb{R}).$$

Therefore it is possible to modify j_M, keeping its domain $T_{M'}$ and target T_M, to obtain a second admissible isomorphism $j'_M : T_{M'} \to T_M$, such that $j'_M(\gamma')$ represents the element in $\mathfrak{D}(M^0_\gamma, M; \mathbb{R})$ different from that represented by $j_M(\gamma')$. The derivation of (6.14.12) does not depend on the choice of j_M, except for that it should satisfy the mild assumptions made in §6.8 when it was chosen, but we can easily arrange so that these assumptions are still true for the modified j'_M. Therefore (6.14.12) is still valid when we replace j_M by j'_M. However, the right hand side of it would be multiplied by a constant not equal to 1 (in fact -1 by the claim we proved in the beginning of the proof) because of the condition †. We conclude that both sides of (6.14.12) are equal to zero, as desired. $\square$

Now by Fourier analysis on the finite abelian groups $\mathfrak{K}(I_0/\mathbb{Q})^D = \mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$, (cf.

[38], [40], [32]), from the previous corollary we deduce

$$(6.14.13) \qquad \mathscr{C}_M \operatorname{Tr}'_M = -4p^*\tau(M)k(M)k(G)^{-1}\sum_{\gamma_0,\gamma_1}\bar{\iota}^M(\gamma_0)^{-1}\bar{v}(I_0)^{-1}e(I_{0,\mathbb{R}})\mathscr{D}(\gamma_1)$$

$$SO_{\gamma_{0,\mathrm{GL}}}(k_a)L_M(\gamma_0)O_{\gamma_1}(f_M^{p,\infty})O_{\gamma_{1,\mathrm{SO}}}(1_{M^{\mathrm{SO}}})\left[\tau(M)^{-1}\tau(I_0)\left|\ker(\ker^1(\mathbb{Q},I_0)\to\ker^1(\mathbb{Q},M))\right|\right],$$

where

- γ_0 runs through a fixed set of representatives of the stable conjugacy classes in $M(\mathbb{Q})$ that are elliptic over $\mathbb{R}$ and good at ∞.
- $I_0 := M^0_{\gamma_0}$.
- γ_1 runs through the subset of $\mathfrak{D}(I_0, M; \mathbb{A})$ consisting of elements whose images in $\mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$ are trivial. Each such γ_1 determines a conjugacy class in $M(\mathbb{A})$ which we also denote by γ_1.
- The number $\left[\tau(M)^{-1}\tau(I_0)\left|\ker(\ker^1(\mathbb{Q},I_0)\to\ker^1(\mathbb{Q},M))\right|\right]$ is equal to the cardinality of $|\mathfrak{K}(I_0/\mathbb{Q})|$.

The final major step needed to compute Tr'_M is to apply the Base Change Fundamental Lemma to relate $\mathrm{SO}_{\gamma_{0,\mathrm{GL}}}(k_a)$ to the twisted orbital integrals in Kottwitz's point counting formula. We only need this result for $\mathbb{G}_m$, in which case it is trivial, and for GL_2, in which case it was initially proved by Langlands [52]. For an account of the theory for GL_n cf. [6] and for the proof in the general case cf. [10].

Observe that the function $k_a \in \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p))$ defined in (6.14.4) is equal to the image under the base change morphism

$$\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_{p^a})) \to \mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_p))$$

(associated to the natural L-embedding ${}^L M^{\mathrm{GL}} \hookrightarrow {}^L(\mathrm{Res}_{\mathbb{Q}_p}^{\mathbb{Q}_{p^a}} M^{\mathrm{GL}}))$ of the element $p^{-a/2}\phi_a^{M_h}$, resp. $-\phi_a^{M_h}$, resp. $-\phi_a^{M_h} \otimes 1$, in $\mathcal{H}^{\mathrm{ur}}(M^{\mathrm{GL}}(\mathbb{Q}_{p^a}))$, where $\phi_a^{M_h}$ is defined in Definition 2.5.9, when $M = M_1$, resp. M_2, resp. M_{12}. Here when $M = M_{12}$ we have $M^{\mathrm{GL}} = M_h \times \mathbb{G}_m$, and we write $-\phi_a^{M_h} \otimes 1$ corresponding to this decomposition, where 1 is the unit of $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_m(\mathbb{Q}_{p^a}))$. Hence by the Base Change Fundamental Lemma, we have, for any semi-simple conjugacy class (=stable conjugacy class) $\gamma_{0,\mathrm{GL}}$ in $M^{\mathrm{GL}}(\mathbb{Q})$,

(6.14.14)

$$SO_{\gamma_{0,\mathrm{GL}}}(k_a) = \begin{cases} -\sum_\delta e(\delta) TO_\delta(\phi_a^{M_h}), & M = M_2 \\ -p^{-a/2} \sum_\delta e(\delta) TO_\delta(\phi_a^{M_h}), & M = M_1 \\ -\sum_\delta e(\delta) TO_\delta(\phi_a^{M_h}) 1_{\mathbb{Z}_p^\times}(y), & M = M_{12},\ \gamma_{0,\mathrm{GL}} = (x,y) \in M_h \times \mathbb{G}_m \end{cases}$$

where δ runs through the σ-conjugacy classes in $M_h(\mathbb{Q}_{p^a})$ such that it has norm the M_h-component of γ_{GL}, and $e(\delta)$ denotes the Kottwitz sign of the twisted centralizer of δ, which is a reductive group over $\mathbb{Q}_p$. (In fact the summation is either empty or over a singleton, which can be checked directly, or follows from [6].)

The next lemma is sometimes called prestabilization in the literature.

Lemma 6.14.7. *Let $F(x,y)$ be a $\mathbb{C}$-valued function on the set of compatible pairs (x,y) of a stable conjugacy class x in $M(\mathbb{Q})$ and a conjugacy class y in $M(\mathbb{A})$. Then we have*

$$\sum_\gamma \iota^M(\gamma)^{-1} F(\gamma,\gamma) = \sum_{\gamma_0,\gamma_1} \bar{\iota}^M(\gamma_0)^{-1} \left|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M))\right| F(\gamma_0,\gamma_1),$$

where on the LHS γ runs through the conjugacy classes in $M(\mathbb{Q})$ which are $\mathbb{R}$-elliptic, and on the RHS γ_0 runs through an arbitrary set of representatives of the stable conjugacy classes in $M(\mathbb{Q})$ that are $\mathbb{R}$-elliptic, and γ_1 runs through the subset of $\mathfrak{D}(I_0, M; \mathbb{A})$ consisting of elements whose images in $\mathfrak{E}(I_0, M; \mathbb{A}/\mathbb{Q})$ are trivial. Here we have denoted $I_0 := M_{\gamma_0}^0$.

Moreover, we may restrict the summation on the LHS to only those γ good at ∞, and restrict the summation on the RHS to only those γ_0 good at ∞, and still get equality.

Proof. The multiplicity of a $M(\mathbb{Q})$-conjugacy class γ appearing in the set $\mathfrak{D}(I_0, M; \mathbb{Q})$ times $\bar{\iota}^M(\gamma_0)^{-1}$ is equal to $\iota^M(\gamma)^{-1}$. The fibers of the map $\mathfrak{D}(I_0, M; \mathbb{Q}) \to \mathfrak{D}(I_0, M, \mathbb{A})$ all have size $\left|\ker(\ker^1(\mathbb{Q}, I_0) \to \ker^1(\mathbb{Q}, M))\right|$. $\square$

We are now ready to prove Theorem 6.5.3.

Proof of Theorem 6.5.3. By (6.14.13) and Lemma 6.14.7, we have

$$\mathscr{C}_M \operatorname{Tr}'_M = -4p^* k(M) k(G)^{-1} \sum_{\gamma} \iota^M(\gamma)^{-1} \left[\bar{v}(M^0_\gamma)^{-1} e(M^0_{\gamma,\mathbb{R}}) \mathscr{D}(M^0_\gamma) \tau(M^0_\gamma)\right] SO_{\gamma_{\mathrm{GL}}}(k_a) L_M(\gamma) O_\gamma(f_M^{p,\infty}) O_{\gamma_{\mathrm{SO}}}(1_{M^{\mathrm{SO}}}), \tag{6.14.15}$$

where γ runs through conjugacy classes in $M(\mathbb{Q})$ that are elliptic over $\mathbb{R}$ and good at ∞. By Harder's formula (cf. [18, §7.10]), we have

$$\chi(M^0_\gamma) = \bar{v}(M^0_\gamma)^{-1} e(M^0_{\gamma,\mathbb{R}}) \mathscr{D}(M^0_\gamma) \tau(M^0_\gamma). \tag{6.14.16}$$

Hence from (6.14.14) (6.14.15) (6.14.16) we have

$$\mathscr{C}_M \operatorname{Tr}'_M = 4p^{**} k(M) k(G)^{-1} \sum_{\gamma,\delta} \iota^M(\gamma)^{-1} \chi(M^0_\gamma) e(\delta) TO_\delta(\phi_a^{M_h}) L_M(\gamma) O_\gamma(f_M^{p,\infty}) O_{\gamma_L}(1_{M_l}), \tag{6.14.17}$$

where γ_L denotes the component of γ in M_l under the decomposition $M = M_h \times M_l$ (which only differs from the decomposition $M = M^{\mathrm{GL}} \times M^{\mathrm{SO}}$ when $M = M_{12}$), and 1_{M_l} denotes the unit in $\mathcal{H}^{\mathrm{ur}}(M_l(\mathbb{Q}_p))$. The number p^{**} denotes p^* when $M = M_2$ or M_{12} and denotes $p^{-a/2}p^*$ when $M = M_1$. To finish the proof we discuss in cases.

Case $M = M_{12}$.

In Definition 2.6.1, δ runs through those elements in $\mathbb{Q}_{p^a}^\times$ with norm γ_0 and such that the Kottwitz invariant of δ in $\pi_1(M_h)_{\Gamma_p} = X_*(\mathbb{G}_m) = \mathbb{Z}$ is equal to the image of $-\mu$. The last condition is equivalent to requiring that $v_p(\delta) = -1$, which is a necessary (and also sufficient) condition for $TO_\delta(\phi_a^{M_h}) \neq 0$. Hence we may drop this condition in the summation in Definition 2.6.1. Every term $c(\gamma_0, \gamma, \delta)$ is easily computed to be 2^{-1} (with $c_1 = \mathrm{vol}(\mathbb{G}_m(\mathbb{R})/\mathbb{G}_m(\mathbb{R})^0)^{-1} = 2^{-1}, c_2 = 1$). On the other hand in (6.14.17) every term $e(\delta)$ is 1. Comparing Definition 2.6.1 and (6.14.17), we see that it suffices to prove that for $\gamma = \gamma_h\gamma_L$ contributing to (6.14.17), we have

$$2^{-1}\delta^{1/2}_{P_{12}(\mathbb{Q}_p)}(\gamma) = 4p^* k(M)k(G)^{-1}\chi(M_{h,\gamma}). \tag{6.14.18}$$

We have

$$\chi(M_{h,\gamma}) = \chi(\mathbb{G}_m) = 2^{-1} \text{ by Harder's formula}$$

$$k(M) = 2^{m-3}, k(G) = 2^{m-1} \text{ by Proposition 6.2.3}$$

Moreover, if $\gamma = \gamma_h\gamma_L$ contributes then $v_p(\gamma_h) = -a$ (because δ should exist) and γ_L lies in a conjugate of $M_l(\mathbb{Z}_p)$, so we can compute

$$\delta_{P_{12}(\mathbb{Q}_p)}(\gamma_L\gamma_0) = \prod_{\alpha\in\Phi^+-\Phi^+_M} |\alpha(\gamma_L\gamma_0)|_p = |\gamma_0|_p^{2m-1} = p^{(d-2)a} = (p^*)^2.$$

Here the contributing roots are $\epsilon_1, \epsilon_1 \pm \epsilon_j, j \geq 2$. The equality (6.14.18) follows.

Case $M = M_1$.

First we claim that if $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ is semi-simple and $\mathbb{R}$-elliptic, then

$$c_2(\gamma_0) = \tau(\mathrm{GL}_{2,\gamma_0}) = 1.$$

In particular[26],

$$c(\gamma_0) = c_1(\gamma_0)c_2(\gamma_0) = \mathrm{vol}(A_{\mathrm{GL}_2}(\mathbb{R})^0 \backslash \overline{\mathrm{GL}_{2,\gamma_0}}(\mathbb{R}))^{-1}.$$

In fact, abbreviate $I_0 := \mathrm{GL}_{2,\gamma_0}$. If $I_0 = \mathrm{GL}_2$, then $\tau(I_0) = c(\gamma_0) = 1$. Otherwise $I_0 = T$ is a maximal torus of GL_2 that is elliptic over $\mathbb{R}$. Assume this. We first show that $c_2(\gamma_0) = 1$. Observe that $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ for some imaginary quadratic field F, so $\mathbf{H}^1(\mathbb{Q}, T) = 0$ by Shapiro's Lemma and Hilbert 90. Hence $c_2(\gamma_0) = 1$. Now by [37] and [39], we have

$$\tau(T)c_2(\gamma_0) = \tau(T)\left|\ker^1(\mathbb{Q}, T)\right| = \tau(I_0)\left|\ker^1(\Gamma, \hat{T})\right| = \left|\pi_0(\hat{T}^\Gamma)\right|.$$

Thus it suffices to show that $\hat{T}^\Gamma$ is connected. We have seen in the proof of Lemma 2.5.6 that $\hat{T}^{\Gamma_\infty} \subset Z(\hat{\mathrm{GL}}_2)$. On the other hand $Z(\hat{\mathrm{GL}}_2) \subset \hat{T}^\Gamma$. Hence $\hat{T}^\Gamma = Z(\hat{\mathrm{GL}}_2) = \mathbb{C}^\times$, as desired.

We continue to consider such $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$. By Harder's formula we have

$$\chi(I_0) = e(I_{0,\mathbb{R}})\bar{v}^{-1}(I_0)\tau(I_0)\left|\mathscr{D}(I_0)\right|.$$

Note that since $\mathrm{GL}_{2,\gamma,\mathbb{R}}$ is either $\mathrm{GL}_{2,\mathbb{R}}$ or an elliptic maximal torus of $\mathrm{GL}_{2,\mathbb{R}}$, we have $e(I_0, \mathbb{R}) = |\mathscr{D}(I_0)| = 1$. Hence

$$\chi(I_0) = e(\overline{I_0})\,\mathrm{vol}(\overline{I_0}/A_{\mathrm{GL}_2}(\mathbb{R})^0)^{-1}\tau(I_0)$$

where $\overline{\mathrm{GL}_{2,\gamma}}$ is the compact-mod-center inner form over $\mathbb{R}$ of $\mathrm{GL}_{2,\gamma}$.

If $\delta \in G(\mathbb{Q}_{p^a})$ has norm stably conjugate to some $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ and γ_0 is good at p (i.e. its determinant has valuation $-a$), then we have

$$e(\delta) = e(\overline{I_0}).$$

[26] This also follows from the formula for c on p. 174 of [40], the fact that $\tau(\mathrm{GL}_2) = 1$, and Lemma 2.5.6

This follows in general from the fact that the Kottwitz invariant automatically vanishes (and the observation that for any place $v \neq p$ or ∞, $e(I_{0,v}) = 1$ since $I_{0,v}$ is either a torus or $\mathrm{GL}_{2,v}$).

So far we have shown that for $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ we have $c(\gamma_0) = e(\delta)\chi(I_0)$, for any $\delta \in M_h(\mathbb{Q}_{p^a})$ with norm γ_0. Moreover, if $\delta \in M_h(\mathbb{Q}_{p^a})$ is such that $TO_\delta(\phi_a^{M_h}) \neq 0$, then necessarily $v_p(\det\delta) = -1$, and it follows easily that the Kottwitz invariant of δ in $\pi_1(M_h)_{\Gamma_p} \cong \mathbb{Z}$ is equal to the image of $-\mu$. It remains to show that

$$\delta_{P_1(\mathbb{Q}_p)}(\gamma_h\gamma_L)^{1/2} = 4p^{**}k(M)k(G)^{-1},$$

for any $\gamma = \gamma_h\gamma_L$ contributing to (6.14.17). We have $k(M) = 2^{m-3}, k(G) = 2^{m-1}$. For $\gamma = \gamma_h\gamma_L$ contributing, we have $v_p(\det\gamma_h) = -a$ (because δ should exist) and γ_L is in a conjugate of $M_l(\mathbb{Z}_p)$. We can compute

$$\delta_{P_1(\mathbb{Q}_p)}(\gamma) = \prod_{\alpha\in\Phi^+-\Phi_M^+} |\alpha(\gamma)|_p = |\det(\gamma_{\mathrm{GL}_2})|_p^{2m-2} = p^{(d-3)a} = (p^{**})^2.$$

Here the contributing roots are $\epsilon_1, \epsilon_2, \epsilon_1+\epsilon_2, \epsilon_1\pm\epsilon_j, \epsilon_2\pm\epsilon_j, j\geq 3$.

Case $M = M_2$ (odd case).

Similarly to the case $M = M_{12}$, we reduce to proving

$$2^{-1}\delta_{P_{12}(\mathbb{Q}_p)}^{1/2}(\gamma) = 2^{-1}4p^{*}k(M)k(G)^{-1}\chi(M_{h,\gamma}).$$

The extra factor 2^{-1} on the RHS in comparison to (6.14.18) comes from the fact that $\mathscr{C}_M = 2$ for $M = M_2$. We have

$$\chi(M_{h,\gamma}) = \chi(\mathbb{G}_m) = 2^{-1} \quad \text{by Harder's formula}$$

$$k(M) = 2^{m-2}, k(G) = 2^{m-1} \text{ by Proposition 6.2.3}$$

Also as in the M_{12} case, if $\gamma = \gamma_h \gamma_L$ contributes then

$$\delta_{P_2(\mathbb{Q}_p)}(\gamma_L\gamma_0) = \prod_{\alpha \in \Phi^+ - \Phi_M^+} |\alpha(\gamma_L\gamma_0)|_p = |\gamma_0|_p^{2m-1} = p^{(d-2)a} = (p^*)^2.$$

Here the contributing roots are $\epsilon_1, \epsilon_1 \pm \epsilon_j, j \geq 2$. □

Remark 6.14.8. The dichotomy of $\mathscr{C}_M$ matches the dichotomy of $k(M)$.

6.15 Main computation, odd case vanishing

Assume we are in the odd case.

Consider an arbitrary cuspidal Levi subgroup M^* of G^*. Then M^* is of the form $\mathbb{G}_m^r \times \mathrm{GL}_2^t \times (M^*)^{\mathrm{SO}}$ where $(M^*)^{\mathrm{SO}}$ is a (split) odd special orthogonal group. The bi-elliptic G^*-endoscopic data of M^* are classified by tuples $(A, B, d^+ = 2n^+ + 1, d^- = 2n^- + 1)$, where A is a subset of $[r]$ and B is a subset of $[t]$, and $n^+ + n^-$ equals the rank of $(M^*)^{\mathrm{SO}}$. Here for an integer $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, 2, \cdots n\}$. A bi-elliptic G^*-endoscopic datum induces an elliptic endoscopic datum $(H, {}^L H, s, \eta)$ for G^*, and A (resp. B) records the positions of the $\mathbb{G}_m$'s (resp. GL_2's) that go to the positive part of H. Now $(H, {}^L H, s, \eta)$ can also be viewed as an endoscopic datum for G, and in particular we have the test function f^H on H (assuming $s \neq 1$), a fixed pair $(j : T_H \to T_G, B_{G,H})$, and a normalization for transfer factors between H and G at all finite places, as before.

Theorem 6.15.1. *Suppose M^* does not transfer to G. Then*

$$\sum_{(M', s_M, \eta_M) \in \mathscr{E}_G(M^*)} |\mathrm{Out}_G(M', s_M, \eta_M)|^{-1} \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H) = 0.$$

Proof. Denote the LHS by Tr'. In the proof we use the symbol Const. to denote a quantity that is constant in a summation. By assumption, at least one of the following is true:

- $rt > 0$.
- $r \geq 3$.
- $t \geq 2$.

Firstly we may assume that M^* transfers to G at all finite places $l \neq p$, because otherwise it is easy to see that $\mathrm{Tr}' = 0$, for example by [59, Lemma 6.3.5 (ii)].

Then we may argue in the same way as the proof of Theorem 6.5.2, to see that Tr' is a linear combination of terms $\mathrm{Tr}'(\mathfrak{e})$, parametrized by $\mathfrak{e} \in \mathscr{E}(M)$, and that each $\mathrm{Tr}'(\mathfrak{e})$ is a linear combination of terms $R_i,\ 1 \leq i \leq r,\ T_j,\ 1 \leq j \leq t$, and S, defined as follows:

$$R_i = R_i(\gamma') := \sum_{A,B} \epsilon_i(A)\omega(A)\,\text{♫}(A,B)\epsilon_R(j_{M^*}(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}),\Theta^H_{\phi_{\mathbb{V}^*}}).$$

$$T_j = T_j(\gamma') := \sum_{A,B} \upsilon_j(B)\omega(A)\,\text{♫}(A,B)\epsilon_R(j_{M^*}(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}),\Theta^H_{\phi_{\mathbb{V}^*}}).$$

$$S = S(\gamma') := \sum_{A,B} \omega(A)\,\text{♫}(A,B)\epsilon_R(j_{M^*}(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}),\Theta^H_{\phi_{\mathbb{V}^*}}).$$

Here γ' runs through the $\mathbb{R}$-elliptic stable conjugacy classes in $M'(\mathbb{Q})$.

Our goal is to prove that for a fixed γ', $R_i = T_j = S = 0$. By continuity, we may assume that the r coordinates of γ' in $\mathbb{G}_m^r$ and the t coordinates in GL_2^t are all distinct.

Explanation of notations: j_{M^*} is an admissible isomorphism from an elliptic maximal torus $T_{M'}$ of M' to one T_{M^*} of M^*. H is the endoscopic datum for G determined by $\mathfrak{e}$ and (A,B). The terms

$$\epsilon_R(j_{M^*}(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}),\Theta^H_{\phi_{\mathbb{V}^*}}) \tag{6.15.1}$$

are computed using the real root system R inside the root system for (T_{M^*}, G^*) and its subsystem R_H defined by H [27] and using a choice of Borel B_0 of $G^*_{\mathbb{C}}$ containing $T_{M^*,\mathbb{C}}$. Note that when we chose B_0 in the discussion above Remark 6.8.1, we required it to satisfy conditions (1) and (2). In the current case we have also numbered the r factors of $\mathbb{G}_m^r$ as $1, 2, \cdots, r$, (so that we are summing over $A \subset [r]$ rather than over subsets A of the set of the r factors of $\mathbb{G}_m^r$,) and we require B_0 to satisfy the analogue of conditions (1) (2) loc. cit. However, we obviously have the liberty of numbering the r factors of $\mathbb{G}_m^r$ and re-trivialize $\mathbb{G}_m$ by the inversion map $\mathbb{G}_m \to \mathbb{G}_m$, in a way that depends on γ'. In fact, we may and will assume that the conditions are satisfied:

1. There exists $r' \in [r] \cup \{0\}$ such that for $i \in [r]$, the i-th coordinate of γ' in $\mathbb{G}_m^r$ is negative if and only if $i > r'$.

2. For all $1 \leq i < i' \leq r'$, or $r' + 1 \leq i < i' \leq r$, we have

$$\frac{\text{the } i\text{-th coordinate of } \gamma'^{-1}}{\text{the } i'\text{-th coordinate of } \gamma'^{-1}} \in]0,1[, \text{ the } i\text{-th coordinate of } \gamma'^{-1} \in]0,1[.$$

We will denote the above product (6.15.1) simply as $\Phi(\gamma', A)$, noting that it does not depend on B.

Define

$$\epsilon_i(A) = \begin{cases} 1, & i \in A \\ -1, & i \notin A \end{cases}, \quad v_j(B) = \begin{cases} 1, & j \in B \\ -1, & j \notin B \end{cases}.$$

We will use the familiar notations n^+, n^-, m^+, m^- to denote the ranks of the positive and negative parts of $(M')^{\mathrm{SO}}$ and H. Thus we have

$$m^+ = n^+ + |A| + 2|B|.$$

[27] The way to view R_H as a root system on $T_{M'}$ is through the embedding $M' \hookrightarrow H$ determined by (A, B)

$$m^- = n^- + |A^c| + 2|B^c|.$$

The sign $\omega(A)$ is up to a multiplicative constant the analogue of $\det(\omega_{0,*})$ in §6.8. Define $\omega_0(A) :=$ the sign of the element $\sigma(A)$ in $\mathfrak{S}_r$ sending

$$(1, 2, \cdots, r) \mapsto (b_1, b_2, \cdots, b_{|A^c|}, a_1, a_2 \cdots, a_{|A|}),$$

where we write $A = \{a_1 < a_2 < \cdots < a_{|A|}\}$ and $A^c = \{b_1 < b_2 < \cdots < b_{|A|^c}\}$. Then

$$\omega(A) := \omega_0(A)(-1)^{|A|n^-}.$$

♫(A, B) is up to a multiplicative constant (which is independent of A, B) the analogue of ♫(A) in §6.9. Namely, in Definition 6.9.1 we replace the parameter A by the parameter (A, B). We emphasize that the non-archimedean transfer factors in this modified version of Definition 6.9.1 are still between M' and M, where M is a local transfer of M^* to G, rather than between M' and M^*. It can be proved similarly as Proposition 6.9.3 that ♫(A, B) is up to a multiplicative constant the product of $(-1)^{q(H)}$ and the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization. We compute

$$q(H) = \frac{m^+(m^+ + 1) + m^-(m^- + 1)}{2} =$$

$$= \frac{(n^+ + |A| + 2|B|)(n^+ + |A| + 2|B| + 1) + (n^- + |A^c| + 2|B^c|) + (n^- + |A^c| + 2|B^c| + 1)}{2}$$

$$\equiv \text{Const.} + (m+1)(|A| + 2|B|) \equiv \text{Const.} + (m+1)|A| \mod 2.$$

As in the proof of Proposition 6.9.3, the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is

$$(-1)^{\lceil m^+/2 \rceil + 1} = (-1)^{\lceil \frac{n^+ + |A| + 2|B|}{2} \rceil + 1}.$$

When n^+ is even, the above is equal to Const.$(-1)^{\lceil |A|/2 \rceil + |B|}$. When n^+ is odd, the above is equal to Const.$(-1)^{\lfloor |A|/2 \rfloor + |B|}$. In any case, taking into account of the fact $m = n^+ + n^- + r$, we can simplify

$$\omega(A)\,\text{♫}(A,B) = \text{Const.}\,\omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor + |B|}.$$

Hence we have, up to multiplicative constants,

$$R_i = \sum_{A,B} \epsilon_i(A)\omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor + |B|}\Phi(\gamma', A)$$

$$T_j = \sum_{A,B} \upsilon_j(B)\omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor + |B|}\Phi(\gamma', A)$$

$$S = \sum_{A,B} \omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor + |B|}\Phi(\gamma', A).$$

We first treat the case $t \geq 2$, which is the easiest. In this case we have, for any fixed A,

$$\sum_B (-1)^{|B|} = 0,$$

and

$$\sum_B \upsilon_j(B)(-1)^{|B|} =$$

$$= \sum_{k=0}^{t} (-1)^k \big[\#\left\{B \,|\, |B| = k, j \in B\right\} - \#\left\{B \,|\, |B| = k, j \notin B\right\}\big]$$

$$= \sum_{k=0}^{t} (-1)^k \left[\binom{t-1}{k-1} - \binom{t-1}{k} \right] = 0.$$

(Note that the above is not true when $t = 1$.) Hence we have $R_i = T_j = S = 0$ in this case.

Before treating the other cases, we compute that

$$\omega_0(A)(-1)^{r|A| + \lfloor |A|/2 \rfloor + |B|}\omega_0(A^c)(-1)^{r|A^c| + \lfloor |A^c|/2 \rfloor + |B|} = (-1)^{\lceil r/2 \rceil}, \tag{6.15.2}$$

using the observation that

$$\omega_0(A)\omega_0 A^c = (-1)^{|A||A^c|}.$$

Now suppose $rt > 0$ and $r = 1$ or 2. Again we have $\sum_B (-1)^{|B|} = 0$, so $R_i = S = 0$. To show $T_j = 0$, observe that $\Phi(\gamma', A) = \Phi(\gamma', A^c)$, so it suffices to show that (6.15.2) is -1. But this is true for $r = 1, 2$.

Finally we treat the case $r \geq 3$, which is the most complicated case. We need a computation that is quite similar to [60, pp. 1698-1699]. In particular we need to use the result of Herb [23]. In the following we will view γ' and B as being fixed, and let A vary.

Recall the definition of $\Phi(\gamma', A)$. We have

$$A_{M'} = A_{M^*} = \mathbb{G}_m^r \times \mathbb{G}_m^t,$$

where the $\mathbb{G}_m^t$ is the product of the centers of the t copies of GL_2's. Write the natural characters on $\mathbb{G}_m^r$ as $\epsilon_1, \cdots, \epsilon_r$, and the natural characters on $\mathbb{G}_m^t$ as $\alpha_1, \cdots, \alpha_t$. Fix γ'. Define

$$I^+ := \{i \in [r] | \epsilon_i(\gamma') > 0\}.$$

$$I^- := [r] - I^+.$$

$$A^+ := A \cap I^+$$

$$A^- := A \cap I^-$$

$$A^{c,+} := A^c \cap I^+$$

$$A^{c,-} := A^c \cap I^-.$$

(By the condition (1) on γ' and the ordering of the r factors of $\mathbb{G}_m^r$ beneath (6.15.1), we know that $I^+ = \{i \in [r] | i \leq r'\}$.) Then the real root system involved in the definition of

$\Phi(\gamma', A)$ is of type

$$B_{|A^+|} \times B_{|A^{c,+}|} \times D_{|A^-|} \times D_{|A^{c,-}|} \times A_1^{\times t},$$

where $B_{|A^+|}$ consists of the roots

$$\epsilon_i,\ \epsilon_i \pm \epsilon_j,\ i, j \in A^+, i \neq j$$

and $D_{|A^-|}$ consists of the roots

$$\epsilon_i \pm \epsilon_j,\ i, j \in A^-, i \neq j,$$

and similarly for $B_{|A^{c,+}|}$ and $D_{|A^{c,-}|}$, and $A_1^{\times t}$ consists of the roots[28]

$$\pm\alpha_1, \cdots, \pm\alpha_t.$$

In particular we see that for γ' to contribute it is necessary that $|A^-|$ and $|A^{c,-}|$ (and a fortiori $|I^-|$) are even. This is equivalent to the condition that $\gamma' \in H_{A,B}(\mathbb{R})^0$, because the Levi subgroup of H in question, inside which the image of γ' is elliptic, is of the form $\mathbb{G}_m^r \times \mathrm{GL}_2^t \times$ a compact SO $\times$ a compact SO. (We have written $H_{A,B}$ for H emphasizing the dependence on A, B.)

Denote by $R_{A,\gamma'}$ the above root system of type $B_{|A^+|} \times B_{|A^{c,+}|} \times D_{|A^-|} \times D_{|A^{c,-}|} \times A_1^{\times t}$.

Then

$$\Phi(\gamma', A) = \sum_{\omega \in \Omega} C(\gamma', \omega) \epsilon_{R_H}(\gamma'^{-1}) n_A(\gamma', \omega B_0),$$

where the coefficients $C(\gamma', \omega)$ are independent of A, and n_A is computed using the root

[28] Note that if ϵ_1, ϵ_2 denote the two standard characters on the diagonal torus in GL_2, and identified with two characters on an elliptic maximal torus of $\mathrm{GL}_{2,\mathbb{R}}$, then with respect to the real structure of the latter $\pm(\epsilon_1 + \epsilon_2)$ are the only real characters among $\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2, -\epsilon_1 \pm \epsilon_2$.

system $R_{A,\gamma'}$:

$$n_A(\gamma', \omega B_0) := \bar{c}_{R_{A,\gamma'}}(x, \wp(\omega\lambda_{B_0} + \omega\rho_{B_0})).$$

For us, it is only important to know that

$$x \in X_*(A_{M^*})_{\mathbb{R}}$$

is such that $\gamma'^{-1} \in \exp(x)T_{M^*}(\mathbb{R})_1$, and that

$$\chi = \wp(\omega\lambda_{B_0} + \omega\rho_{B_0}) \in X^*(A_{M^*})_{\mathbb{R}}$$

is independent of A.

By the assumptions (1)(2) beneath (6.15.1), we easily compute

$$\epsilon_{R_H}(\gamma'^{-1}) = (-1)^{2^{-1}\left[\left|A^+\right|(\left|A^+\right|+1)+\left|A^{c,+}\right|(\left|A^{c,+}\right|+1)+\left|A^-\right|(\left|A^-\right|-1)+\left|A^{c,-}\right|(\left|A^{c,-}\right|-1)\right]}.$$

Since $|A^-|$ and $|A^{c,-}|$ are even, we have

$$\epsilon_{R_H}(\gamma'^{-1}) = \text{Const.}(-1)^{(r+1)|A|}.$$

Thus to show that $R_i = T_j = S = 0$, it suffices to show that the following two are zero, where the sums run over $A \subset [r]$ such that A^- and $A^{c,-}$ have even cardinality:

$$M_i := \sum_A \epsilon_i(A)\omega_0(A)(-1)^{|A|+\lceil |A|/2\rceil}\bar{c}_{R_{A,\gamma'}}(x, \chi).$$

$$N := \sum_A \omega_0(A)(-1)^{|A|+\lceil |A|/2\rceil}\bar{c}_{R_{A,\gamma'}}(x, \chi).$$

More precisely, proving $N = 0$ already shows that $T_j = S = 0$.

We now need to apply Herb's formula for $\bar{c}_{R_{A,\gamma'}}$. We will recall and follow the notation and definitions of [60, pp. 1698-1699]. Note that in loc. cit. one concerns root systems of types C and D, whereas in our case we need to consider root systems of types B and D, but the formulas for type B and type C root systems are identical. cf. [23].

To avoid confusion of notation, in the following we write $\mathbb{B}$ and $\mathbb{D}$ when we want to indicate the type of a root system.

For $a, b \in \mathbb{R}$, we define

$$c_1(a) := \begin{cases} 1, & a > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$c_{2,\mathbb{B}}(a,b) := \begin{cases} 1, & 0 < a < b \text{ or } 0 < -b < a \\ 0, & \text{otherwise} \end{cases}$$

$$c_{2,\mathbb{D}}(a,b) := \begin{cases} 1, & a > |b| \\ 0, & \text{otherwise} \end{cases}$$

Our $c_{2,\mathbb{B}}$ is equal to Morel's $c_{2,C}$. Also note that the specific positions of the non-zero areas of the above functions are designed to suit the conditions (1)(2) of γ'^{-1} beneath (6.15.1).[29]

For a finite set I define $\mathcal{P}^0_{\leq 2}(I)$ to be the set of unordered partitions $\{I_z, z \in Z\}$ of I such that all I_z have cardinality 2 or 1 and at most one I_z has cardinality 1. We will simplify notation to write $\mathcal{P}(I)$ for $\mathcal{P}^0_{\leq 2}(I)$. If I has a total order, we can define a sign function

$$\epsilon : \mathcal{P}(I) \to \{\pm 1\}$$

as follows. Given $p = \{I_z | z \in Z\} \in \mathcal{P}(I)$, choose an enumeration $Z \xrightarrow{\sim} \{1, 2, \cdots, |Z|\}$. Thus $p = \left\{I_1, I_2, \cdots, I_{|Z|}\right\}$. Define $\sigma \in \mathfrak{S}(I)$ such that

[29] To compare with [60], our γ'^{-1} is analogous to γ_M in loc. cit.

- For all $1 \leq i < j \leq |Z|$, and for all $s \in \sigma(I_i), s' \in \sigma(I_j)$, we have $s < s'$.
- For all $z \in Z$ such that $|I_z| = 2$, writing $I_z = \{s_1, s_2\}$ with $s_1 < s_2$, we have $\sigma(s_1) < \sigma(s_2)$.

Then we define

$$\epsilon(p) := \mathrm{sgn}(\sigma).$$

The definition does not depend on the choice of the enumeration of Z.

For $\mu \in \mathbb{R}^r$ and J a subset of $[r]$ of cardinality 1 or 2, we make the following definitions. If $J = \{s\}$, define

$$c_{J,\mathbb{B}}(\mu) := c_1(\mu_s)$$

If $J = \{s_1, s_2\}$ with $s_1 < s_2$, define

$$c_{J,\mathbb{B}}(\mu) := c_{2,\mathbb{B}}(\mu_{s_1}, \mu_{s_2})$$

$$c_{J,\mathbb{D}}(\mu) := c_{2,\mathbb{D}}(\mu_{s_1}, \mu_{s_2})$$

Now for $I \subset [r]$ and $p = \{I_z | z \in Z\} \in \mathcal{P}(I)$, define

$$c_{\mathbb{B}}(p, \mu) := \prod_{z \in Z} c_{I_z,\mathbb{B}}(\mu).$$

If in addition $|I|$ is even, define

$$c_{\mathbb{D}}(p, \mu) := \prod_{z \in Z} c_{I_z,\mathbb{D}}(\mu).$$

We can now state Herb's formula. Let $\chi \in X^*(A_{M^*})_{\mathbb{R}}$ and let μ be its projection to $X^*(\mathbb{G}_m^r)_{\mathbb{R}} \cong \mathbb{R}^r$, where we have trivialized $X^*(\mathbb{G}_m^r)$ using the characters $\epsilon_1, \cdots, \epsilon_r$. Let x be related to γ'^{-1} as above. Then Herb's formula states that

$$\bar{c}_{R_{A,\gamma'}}(x,\chi) = \text{Const.} \sum_{p_1^+ \in \mathcal{P}(A^+)} \sum_{p_1^- \in \mathcal{P}(A^-)} \sum_{p_2^+ \in \mathcal{P}(A^{c,+})} \sum_{p_2^- \in \mathcal{P}(A^{c,-})} \epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) \cdot$$

$$\cdot c_{\mathbb{B}}(p_1^+,\mu) c_{\mathbb{B}}(p_2^+,\mu) c_{\mathbb{D}}(p_1^-,\mu) c_{\mathbb{D}}(p_2^-,\mu),$$

where Const. is independent of A.

(To compare with the formula on p. 1699 of [60], note that the root system loc. cit. is of type $C_{|A_1^-|} \times C_{|A_1^+|} \times D_{|A_2^-|} \times D_{|A_2^+|} \times A_1^{\times t}$, whereas our root system is $B_{|A^+|} \times B_{|A^{c,+}|} \times D_{|A^-|} \times D_{|A^{c,-}|} \times A_1^{\times t}$.)

First assume r is odd.

Since $|A^-|$ and $|A^{c,-}|$ must be even, we know that $|A^+|$ and $|A^{c,+}|$ must have different parity. In particular I^+ is non-empty. For $p_1^+ \in \mathcal{P}(A^+)$ and $p_2^+ \in \mathcal{P}(A^{c,+})$, we have $p^+ := p_1^+ \cup p_2^+ \in \mathcal{P}(I^+)$. Also for $p_1^- \in \mathcal{P}(A^-)$ and $p_2^- \in \mathcal{P}(A^{c,-})$, we have $p^- := p_1^- \cup p_2^- \in \mathcal{P}(I^-)$. We also have

$$\omega_0(A)\epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) = \epsilon(p^+)\epsilon(p^-).$$

Conversely, given $p^+ \in \mathcal{P}(I^+)$, write $p^+ = \{I_1, \cdots, I_k\}$ with $k = \lceil |I^+|/2 \rceil \geq 1$, such that $I_1, \cdots, I_{k-1}$ have cardinality 2 and $|I_k| = 1$. To recover A^+ it suffices to choose a subset U of $[k-1]$ and a subset V of $\{k\}$, by defining $A^+ := \bigcup_{j \in U \cup V} I_j$. We have $|A^+| = 2|U| + |V|$. If $s \in I_j$ is a fixed element for some $1 \leq j \leq k-1$, we have $s \in A$ if and only if $j \in U$. If $s \in I_k$, we have $s \in A$ if and only if $k \in V$. Thus, if we define

$$\epsilon_s(p^+, U, V) := \begin{cases} \epsilon_j(U), \ s \in I_j \text{ for some } 1 \leq j \leq k-1 \\ \epsilon_k(V), \ s \in I_k \end{cases}$$

then we have

$$\epsilon_s(A^+) = \epsilon_s(p^+, U, V).$$

(Strictly speaking the above definition of $\epsilon_s(p^+, U, V)$ depends on an enumeration of elements of p^+ if we think of the domain of U as the power set of $[k-1]$ and the domain of V as the power set of $[k]$, but we can make the definition independent by viewing U, V abstractly as suitable subsets of p^+.)

Similarly, given $p^- \in \mathcal{P}(I^-)$, write $p^- = \{I_1, \cdots, I_l\}$ with $l = |I^-|/2$. To recover A^- it suffices to choose a subset W of $[l]$ and define $A^- := \bigcup_{j \in W} I_j$. We have $|A^-| = 2|W|$. For a fixed $s \in I^-$, define

$$\epsilon_s(p^-, W) := \epsilon_{j(s)}(W),$$

where $j(s) \in [l]$ is such that $s \in I_{j(s)}$. Then we have $\epsilon_s(A^-) = \epsilon_s(p^-, W)$.

For $s \in [r]$, we define

$$\epsilon_s(p^+, p^-, U, V, W) := \begin{cases} \epsilon_s(p^+, U, V), \;\; s \in I^+ \\ \epsilon_s(p^-, W), \;\; s \in I^- \end{cases}.$$

In conclusion, we have

$$N = \text{Const.} \sum_{p^+ \in \mathcal{P}(I^+)} \sum_{p^- \in \mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-) c_{\mathbb{B}}(p^+, \mu) c_{\mathbb{D}}(p^-, \mu) \cdot$$

$$\cdot \sum_{U \subset [k-1], V \subset \{k\}, W \subset [l]} (-1)^{2|U| + |V| + 2|W| + \lceil (2|U| + |V| + 2|W|)/2 \rceil}.$$

Hence

$$N = \text{Const.} \sum_{U \subset [k-1], V \subset \{k\}, W \subset [l]} (-1)^{|U| + |W|} = 0,$$

because either $l > 0$, or $r = |I^+| = 2k - 1 \geq 3$ and hence $k - 1 \geq 1$. **At this point we**

have shown that $T_j = S = 0$ when $r \geq 3$ is odd.

Similarly we have

$$M_s = \text{Const.} \sum_{p^+ \in \mathcal{P}(I^+)} \sum_{p^- \in \mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{B}}(p^+, \mu)c_{\mathbb{D}}(p^-, \mu) \cdot$$

$$\cdot \sum_{U \subset [k-1], V \subset \{k\}, W \subset [l]} (-1)^{|U|+|W|} \epsilon_s(p^+, p^-, U, V, W).$$

To show $M_s = 0$, it suffices to prove for any fixed p^+, p^- that

$$L := \sum_{U \subset [k-1], V \subset \{k\}, W \subset [l]} (-1)^{|U|+|W|} \epsilon_s(p^+, p^-, U, V, W)$$

is zero. **We show this when r (assumed to be odd) is ≥ 5.**

If $\epsilon_s(p^+, p^-, U, V, W) = \epsilon_k(V)$, then $L = 0$ because $\sum_{V \subset \{k\}} \epsilon_k(V) = 0$.

Suppose $\epsilon_s(p^+, p^-, U, V, W) = \epsilon_s(p^-, W) = \epsilon_j(W)$ for some $j \in [l]$. Then for $l \geq 2$ we are done because for any fixed $j \in [l]$ we have

$$\sum_{W \subset [l]} \epsilon_j(W)(-1)^{|W|} = 0.$$

For $l \leq 1$, we have $|I^+| = 2k - 1 = r - 2l \geq 5 - 2 = 3$, so $k - 1 \geq 1$, and $L = 0$ because $\sum_{U \subset [k-1]} (-1)^{|U|} = 0$.

Suppose $\epsilon_s(p^+, p^-, U, V, W) = \epsilon_j(U)$ for some $j \in [k-1]$. If $l \geq 1$ we are done because $\sum_{W \subset [l]} (-1)^{|W|} = 0$. Suppose $l = 0$. Then $r = 2k - 1 \geq 5$, and hence $k - 1 \geq 2$. We have

$$\sum_{U \subset [k-1]} (-1)^{|U|} \epsilon_j(U) = 0,$$

and so $L = 0$.

At this point we have shown that $L = 0$ for $r \geq 5$. For odd $r \geq 3$ the only thing left to be proved is that for $r=3$ we have $R_i=0$. Now for $r=3$, the value of (6.15.2) is 1, and we have $\epsilon_i(A)=-\epsilon_i(A^c)$, which implies that $R_i=0$.

Now assume r is even. Then $|I^+|$ and $|I^-|$ are both even. Write $k:=|I^+|/2$ and $l:=|I^-|/2$.

For a set I of even cardinality, we define $\mathcal{P}'(I)$ to be the set of unordered partitions $\{I_z|z\in Z\}$ of I equipped with a marked element in Z, such that exactly two I_z's are singletons and all the rest I_z's have cardinality 2, and for the marked z the corresponding I_z is a singleton.

For $p\in\mathcal{P}'(I)$ and a total order on I, we can still define a sign $\epsilon(p)$, using the definition similar to the sign on $\mathcal{P}(I)$, with the only modification that when enumerating $Z\xrightarrow{\sim}[|Z|]$, we require that the number in $[|Z|]$ corresponding to the marked z is larger than that corresponding to the other $z\in Z$ whose associated I_z is a singleton.

If I is a subset of $[r]$ and $p=\{I_z|z\in Z\}\in\mathcal{P}'(I)$, define

$$c_{\mathbb{B}}(p,\mu):=\prod_{z\in Z}c_{I_z}(\mu).$$

If $|A^+|$ is odd then so is $|A^{c,+}|$, and for $p_1^+\in\mathcal{P}(A^+), p_2^+\in\mathcal{P}(A^{c,+})$, we can define $p^+:=p_1^+\cup p_2^+\in\mathcal{P}'(I^+)$, where the marked singleton in p^+ is defined to be the singleton in p_1^+. Conversely, suppose $k\geq 1$. Given $p^+\in\mathcal{P}'(I^+)$, write $p^+=\{I_1,I_2,\cdots,I_{k-1},I_k,I_\infty\}$, such that I_∞ is the marked singleton and I_k is the other singleton. To recover A^+ it suffices to choose a subset U of $[k-1]$ and define $A^+=\bigcup_{j\in U}I_j\cup I_\infty$. We have $|A^+|=2|U|+1$. For $s\in I^+$, define

$$\epsilon_s(p^+,U):=\begin{cases}\epsilon_{j(s)}(U),\ s\in I_{j(s)}, 1\leq j(s)\leq k-1\\ -1,\ s\in I_k\\ 1,\ s\in I_\infty\end{cases}$$

Then we have $\epsilon_s(p^+, U) = \epsilon_s(A^+)$.

If $|A^+|$ is even, then so is $|A^{c,+}|$. For $p_1^+ \in \mathcal{P}(A^+)$ and $p_2^+ \in \mathcal{P}(A^{c,+})$, we can define $p^+ := p_1^+ \cup p_2^+ \in \mathcal{P}(I^+)$. Conversely, given $p^+ \in \mathcal{P}(I^+)$, write $p^+ = \{I_1, I_2, \cdots, I_k\}$. To recover A^+ it suffices to choose a subset U of $[k]$ and define $A^+ = \bigcup_{j\in U} I_j$. We have $|A^+| = 2\,|U|$. For $s \in I^+$, define

$$\epsilon_s(p^+, U) := \epsilon_{j(s)}(U),$$

where $j(s) \in [k]$ is such that $s \in I_{j(s)}$.

Similarly, for $p_1^- \in \mathcal{P}(A^-)$ and $p_2^- \in \mathcal{P}(A^{c,-})$, we can define $p^- := p_1^- \cup p_2^- \in \mathcal{P}(I^-)$. Conversely, given $p^- \in \mathcal{P}(I^-)$, write $p^- = \{I_1, I_2, \cdots, I_l\}$. To recover A^- it suffices to choose a subset W of $[l]$ and define $A^- = \bigcup_{j\in W} I_j$. We have $|A^-| = 2\,|W|$. For $s \in I^-$, define

$$\epsilon_s(p^-, W) := \epsilon_{j(s)}(W),$$

where $j(s) \in [l]$ is such that $s \in I_{j(s)}$.

For both parity of $|A^+|$, we have

$$\omega_0(A)\epsilon(p_1^+)\epsilon(p_2^+)\epsilon(p_1^-)\epsilon(p_2^-) = \epsilon(p^+)\epsilon(p^-).$$

For $s \in [r]$, define

$$\epsilon_s(p^+, p^-, U, W) := \begin{cases} \epsilon_s(p^+, U), & s \in I^+ \\ \epsilon_s(p^-, W), & s \in I^- \end{cases}.$$

We split

$$N = \sum_{A, |A^-| \text{ even}} \omega_0(A)(-1)^{|A| + \lceil |A|/2 \rceil} \bar{c}_{R_{A,\gamma'}}(x, \chi)$$

into the two sub-sums $N_{(1)}$, $N_{(2)}$ over A, under the condition that $|A^+|$ is odd or even respectively. Similarly we split

$$M_i = \sum_{A,|A^-| \text{ even}} \epsilon_i(A)\omega_0(A)(-1)^{|A|+\lceil |A|/2\rceil}\bar{c}_{R_{A,\gamma'}}(x,\chi)$$

into $M_i = M_{i,(1)} + M_{i,(2)}$. The above discussion shows that

$$\begin{aligned} N_{(1)} = \text{Const.} \sum_{p^+\in\mathcal{P}'(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{B}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu)\cdot \\ \cdot \sum_{U\subset[k-1],W\subset[l]} (-1)^{2|U|+1+2|W|+\lceil(2|U|+1+2|W|)/2\rceil} \\ = \text{Const.} \sum_{U\subset[k-1],W\subset[l]} (-1)^{|U|+|W|}. \end{aligned}$$

This is zero because either $l > 0$ or $r = 2k \geq 4$ and hence $k-1 \geq 1$.

Also

$$\begin{aligned} N_{(2)} = \text{Const.} \sum_{p^+\in\mathcal{P}(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{B}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu)\cdot \\ \cdot \sum_{U\subset[k],W\subset[l]} (-1)^{2|U|+2|W|+\lceil(2|U|+2|W|)/2\rceil} \\ = \text{Const.} \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|}, \end{aligned}$$

which is zero because $kl > 0$.

Similarly

$$\begin{aligned} M_{s,(1)} = \text{Const.} \sum_{p^+\in\mathcal{P}'(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{B}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu)\cdot \\ \cdot \sum_{U\subset[k-1],W\subset[l]} (-1)^{|U|+|W|}\epsilon_s(p^+,p^-,U,W), \end{aligned}$$

and

$$M_{s,(2)} = \text{Const.} \sum_{p^+\in\mathcal{P}(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{B}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu)\cdot$$

$$\cdot \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|}\epsilon_s(p^+,p^-,U,W).$$

Thus to show $M_{s,(1)} = M_{s,(2)} = 0$, it suffices to show that the following $L_{(1)}$ and $L_{(2)}$ are zero for fixed $q^+ \in \mathcal{P}'(I^+), p^+ \in \mathcal{P}(I^+), p^- \in \mathcal{P}(I^-)$:

$$L_{(1)} := \sum_{U\subset[k-1],W\subset[l]} (-1)^{|U|+|W|}\epsilon_s(q^+,p^-,U,W)$$

$$L_{(2)} := \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|}\epsilon_s(p^+,p^-,U,W).$$

We first treat $L_{(2)}$. By symmetry we may assume that $l > 0$ and $s \in I^-$, so that $\epsilon_s(p^+,p^-,U,W) = \epsilon_j(W)$ for some $j \in [l]$. If $k > 0$, then $L_{(2)} = 0$ because $\sum_{U\subset[k]}(-1)^{|U|} = 0$. If $k = 0$, then $r = 2l \geq 4$, so $l \geq 2$. Then we have

$$\sum_{W\subset[l]} (-1)^{|W|}\epsilon_j(W) = 0$$

so $L_{(2)} = 0$.

We now treat $L_{(1)}$. If $s \in I_k$ or I_∞, then $\epsilon_s(q^+,p^-,U,W)$ is independent of (U,W), and

$$L_{(1)} = \text{Const.} \sum_{U\subset[k-1],W\subset[l]} (-1)^{|U|+|W|} = 0,$$

because either $l > 0$ or $r = 2k \geq 4$ and so $k-1 \geq 1$.

For the rest cases, **we assume** $r \geq 6$. Suppose $\epsilon_s(q^+,p^-,U,W) = \epsilon_j(U)$ for some $j \in [k-1]$. If $l > 0$, then $L_{(1)} = 0$ because $\sum_{W\subset[l]}(-1)^{|W|} = 0$. Suppose $l = 0$. Then

$r = 2k \geq 6$, so $k-1 \geq 2$, and we have

$$\sum_{U \subset [k-1]} (-1)^{|U|} \epsilon_j(U) = 0,$$

which implies that $L_{(1)} = 0$.

Suppose $\epsilon_s(q^+, p^-, U, W) = \epsilon_j(W)$ for some $j \in [l]$. If $k \geq 2$, then $L_{(1)} = 0$ because $\sum_{U \subset [k-1]} (-1)^{|U|} = 0$. If $k \leq 1$, then $2l = r - 2k \geq 6 - 2 = 4$, and so

$$\sum_{W \subset [l]} (-1)^{|W|} \epsilon_j(W) = 0,$$

which implies that $L_{(1)} = 0$.

For even r it only remains to show that for $r = 4$ we have $R_i = 0$. This is true because again, for $r = 4$, the value of (6.15.2) is 1, and we have $\epsilon_i(A) = -\epsilon_i(A^c)$, which implies that $R_i = 0$.

□

6.16 Main computation, even case vanishing

Assume we are in the even case. We are to state and prove the analogue of Theorem 6.15.1.

Consider an arbitrary cuspidal Levi subgroup M^* of G^*. Then M^* is of the form $\mathbb{G}_m^r \times \mathrm{GL}_2^t \times (M^*)^{\mathrm{SO}}$ where $(M^*)^{\mathrm{SO}}$ is a quasi-split even special orthogonal group. Cuspidality forces r to be an even number. The bi-elliptic and bi-cuspidal G^*-endoscopic data of M^* are classified by tuples $(A, B, d^+ = 2n^+, d^- = 2n^-, \delta^+, \delta^-)$, where A is a subset of $[r]$ and B is a subset of $[t]$, and $n^+ + n^-$ equals the rank of $(M^*)^{\mathrm{SO}}$. When localized over $\mathbb{R}$, the images of $\delta^{\pm}$ in $\mathbb{R}^{\times}/\mathbb{R}^{\times,2} = \{\pm 1\}$ are determined by $d^{\pm}$ by the cuspidality condition. A bi-elliptic bi-cuspidal G^*-endoscopic datum induces an elliptic cuspidal endoscopic datum $(H, \cdots)$ for G^*, and A (resp. B) records the positions of the $\mathbb{G}_m$'s (resp. GL_2's) that go to the positive part of H. Now $(H, \cdots)$ can also be viewed as an endoscopic datum for G, and in particular we have the test function f^H on H, a fixed pair $(j : T_H \to T_G, B_{G,H})$, and a normalization for transfer factors between H and G at all finite places, as before.

Theorem 6.16.1. *Suppose M^* does not transfer to G. Then*

$$\sum_{(M', s_M, \eta_M) \in \mathscr{E}_G(M^*)} |\mathrm{Out}_G(M', s_M, \eta_M)|^{-1} \tau(G)\tau(H)^{-1} ST^H_{M'}(f^H) = 0.$$

Proof. The proof is quite similar to the odd case. We follow most of the notations established in the proof of Theorem 6.15.1 and indicate the differences.

By assumption at least one of the following is true:

- $rt > 0$
- $r \geq 4$
- $t \geq 2$.

(Recall that r is even).

Again, it suffices to prove the following terms are zero:

$$R_i = \sum_{A,B} \epsilon_i(A)\omega(A) \text{♫}(A, B)\Phi(\gamma', A)$$

$$T_j = \sum_{A,B} \upsilon_j(B)\omega(A) \text{♫}(A, B)\Phi(\gamma', A)$$

$$S = \sum_{A,B} \epsilon_i(A)\omega(A) \text{♫}(A, B)\Phi(\gamma', A).$$

In the above sums A only runs through even subsets of $[r]$, and B runs through subsets of $[t]$.

Here as in the odd case

$$\Phi(\gamma', A) = \epsilon_R(j_{M^*}(\gamma'^{-1}))\epsilon_{R_H}(\gamma'^{-1})\Phi^{G^*}_{M^*}(j_{M^*}(\gamma'^{-1}), \Theta^H_{\phi_{\mathbb{V}^*}}).$$

The Borel B_0 of $G^*_{\mathbb{C}}$ containing $T_{M^*,\mathbb{C}}$ which is chosen implicitly for the above definition is assumed to satisfy the analogues of conditions (1) (2) above Remark 6.8.1. We also assume that the r factors of $\mathbb{G}_m^r$ are numbered in such a way that the following conditions analogous to the odd case are satisfied:

1. There exists $r' \in [r] \cup \{0\}$ such that for $i \in [r]$, the i-th coordinate of γ' in $\mathbb{G}_m^r$ is negative if and only if $i > r'$.

2. For all $1 \leq i < i' \leq r'$, or $r' + 1 \leq i < i' \leq r$, we have

$$\frac{\text{the } i\text{-th coordinate of } \gamma'^{-1}}{\text{the } i'\text{-th coordinate of } \gamma'^{-1}} \in]0, 1[.$$

We define $♫(A, B)$ analogously as in the odd case. To get a closed formula of it, we need to compute the product of $(-1)^{q(H)}$ and the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization. Here we observe that even when d is divisible by 4, we do not need to be specific about which Whittaker datum of G^* and which Whittaker datum of M^* we are considering in order to prove the desired vanishing, because changing these choices would only contribute a sign in front of $♫(A, B)$ that is independent of A and B. We note that $q(H) \equiv 0 \mod 2$ by the cuspidality condition. As in the proof of Proposition 6.9.3 the sign between $\Delta_{j,B_{G,H}}$ and the Whittaker normalization is up to a constant

$$(-1)^{\lfloor m^-/2 \rfloor} = \text{Const.}(-1)^{|B|+|A|/2}.$$

Thus we may define

$$\text{♫}(A, B) := (-1)^{|B|+|A|/2}.$$

Since $|A|$ is even, we have $\omega(A) = \omega_0(A)$. Also we have

$$\omega_0(A) = \omega_0(A^c)(-1)^{|A||A^c|} = \omega_0(A^c)$$

because $|A|$ is even. In particular,

$$\omega(A)\,\text{♫}(A, B)\omega(A^c)\,\text{♫}(A^c, B) = (-1)^{r/2}. \tag{6.16.1}$$

As in the odd case, the case $t \geq 2$ is the easiest. In this case we have

$$\sum_B (-1)^{|B|} = \sum_B \upsilon_j(B)(-1)^{|B|} = 0,$$

so $R_i = T_j = S = 0$.

Assume $t = 1$ and $r = 2$. Then $R_i = S = 0$ because $\sum_B (-1)^{|B|} = 0$. To show $T_j = 0$, we use the fact that (6.16.1) is -1.

Finally we treat the case $r \geq 4$. The discussion in the odd case for $r \geq 3$ needs almost no change to be carried over here, only except that all the sets $I^+, I^-, A^+, A^{c,+}, A^-, A^{c,-}$ have to have even cardinality in the even case, and that the root system $R_{A,\gamma'}$ in question here is of type $D_{|A^+|} \times D_{|A^{c,+}|} \times D_{|A^-|} \times D_{|A^{c,-}|}$. In particular, Herb's formula reads

$$\bar{c}_{R_{A,\gamma'}}(x, \chi) = \text{Const.} \sum_{p_1^+ \in \mathcal{P}(A^+)} \sum_{p_1^- \in \mathcal{P}(A^-)} \sum_{p_2^+ \in \mathcal{P}(A^{c,+})} \sum_{p_2^- \in \mathcal{P}(A^{c,-})} \epsilon(p_1^+)\epsilon(p_1^-)\epsilon(p_2^+)\epsilon(p_2^-) \cdot$$

$$\cdot c_{\mathbb{D}}(p_1^+, \mu) c_{\mathbb{D}}(p_2^+, \mu) c_{\mathbb{D}}(p_1^-, \mu) c_{\mathbb{D}}(p_2^-, \mu).$$

We compute

$$\epsilon_{R_H}(\gamma'^{-1}) = (-1)^{\alpha},$$

where

$$\alpha = 2^{-1}[\left|A^+\right|(\left|A^+\right|-1)+\left|A^-\right|(\left|A^-\right|-1)+\left|A^{c,+}\right|(\left|A^{c,+}\right|-1)+\left|A^{c,-}\right|(\left|A^{c,-}\right|-1)]$$

$$\equiv r/2 \mod 2.$$

Thus $\epsilon_{R_H}(\gamma'^{-1})$ is independent of A. As in the odd case, define

$$M_i := \sum_A \epsilon_i(A)\omega_0(A)(-1)^{|A|/2}\bar{c}_{R_{A,\gamma'}}(x,\chi)$$

$$N = \sum_A \omega_0(A)(-1)^{|A|/2}\bar{c}_{R_{A,\gamma'}}(x,\chi),$$

where A runs through subsets of $[r]$ such that $|A^{\pm}|$ and $|A^{c,\pm}|$ are all even. Then to show $T_j = S = 0$ it suffices to show $N = 0$, and to show $R_i = 0$ it suffices to show $M_i = 0$.

Write $k = |I^+|/2, l = |I^-|/2$. For a fixed $p^+ \in \mathcal{P}(I^+)$, write $p^+ = \{I_1, \cdots, I_k\}$. For a subset U of $[k]$, and for $s \in I_j \subset I^+$, define

$$\epsilon_s(p^+, U) := \epsilon_j(U).$$

Similarly we define $\epsilon_s(p^-, W)$ for W a subset of the indexing set of p^- and $s \in I^-$, and we define

$$\epsilon_s(p^+, p^-, U, W) := \begin{cases} \epsilon_s(p^+, U), \ \ s \in I^+ \\ \epsilon_s(p^-, W), \ \ s \in I^+ \end{cases}$$

as in the odd case.

Herb's formula together with an argument analogous to the odd case implies that

$$N = \sum_{p^+\in\mathcal{P}(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{D}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu) \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|} =$$

$$= \text{Const.} \sum_{U\subset[k],W\subset[l]} (-1)^{|U|+|W|},$$

and that

$$M_s = \sum_{p^+\in\mathcal{P}(I^+)} \sum_{p^-\in\mathcal{P}(I^-)} \epsilon(p^+)\epsilon(p^-)c_{\mathbb{D}}(p^+,\mu)c_{\mathbb{D}}(p^-,\mu) \sum_{U\subset[k],W\subset[l]} \epsilon_s(p^+,p^-,U,W)(-1)^{|U|+|W|}.$$

Since $lk \neq 0$, we have $N = 0$. We now show $M_s = 0$.

Fix p^+, p^-. We are going to show

$$L := \sum_{U\subset[k],W\subset[l]} \epsilon_s(p^+,p^-,U,W)(-1)^{|U|+|W|}$$

is zero. By symmetry we may assume $\epsilon_s(p^+,p^-,U,W) = \epsilon_j(W)$ for some $j \in [l]$. If $k \geq 1$, $L = 0$ because

$$\sum_{U\subset[k]} (-1)^{|U|} = 0.$$

Suppose $k = 0$. Then $r = 2l \geq 4$, so $l \geq 2$. Then $L = 0$ because

$$\sum_{W\subset[l]} \epsilon_j(W)(-1)^{|W|} = 0.$$

This concludes the proof. □

6.17 The main identity

Corollary 6.17.1. *In the even case assume $\mathbb{V}$ is a formal sum $\mathbb{V} = \mathbb{V}_1 + {}^{\mathrm{a}}\mathbb{V}_1$. For $a \in \mathbb{N}$ large enough, we have*

$$\mathrm{Tr}_{M_1}(f^{p,\infty}, a) + \mathrm{Tr}_{M_2}(f^{p,\infty}, a) + \mathrm{Tr}_{M_{12}}(f^{p,\infty}, a) = \sum_{(H,s,\eta)\in\mathscr{E}(G)} \iota(G,H)[ST^H(f^H) - ST^H_e(f^H)].$$

Here $\iota(G,H) := \tau(G)\tau(H)^{-1}\left|\mathrm{Out}(H,s,\eta)\right|^{-1}$, and $ST^H_e := ST^H_H$ as defined in §6.3.

Proof. This follows from Theorems 6.5.3, 6.15.1, 6.16.1, and the following observation which can be verified directly in our case (cf. also [59, Lemma 2.4.2] and the attribution to Kottwitz's unpublished notes loc. cit.):

Let

$$\phi : \coprod_{(H,s,\eta)\in\mathscr{E}(G)} \mathcal{L}(H) \to \mathbb{C}$$

be a finitely supported function, where $\mathcal{L}(H)$ denotes the set of $H(\mathbb{Q})$-conjugacy classes of Levi subgroups of H. Assume that ϕ vanishes on $L \in \mathcal{L}(H)$ whenever L is not cuspidal, for all H. Then we have

$$\sum_{(H,s,\eta)\in\mathscr{E}(G)} \left|\mathrm{Out}(H,s,\eta)\right|^{-1} \sum_{L\in\mathcal{L}(H)} (n^H_L)^{-1}\phi(L) =$$

$$= \sum_{M\in\mathcal{L}(G^*)} (n^{G^*}_M)^{-1} \sum_{(M',s_M,\eta_M)\in\mathscr{E}_{G^*}(M)} \left|\mathrm{Out}_{G^*}(M',s_M,\eta_M)\right|^{-1} \phi(M' \subset H(M',s_M,\eta_M)),$$

where $H(M',s_M,\eta_M)$ is the endoscopic group for G^* (or equivalently G) induced by the G^*-endoscopic datum (M',s_M,η_M) for M, so that M' is a Levi subgroup of $H(M',s_M,\eta_M)$. □

Remark 6.17.2. Strictly speaking ST^H_e has its own definition, and should not have been defined to be ST^H_H. However for the particular form of f^H_∞, we have $ST^H_e(f^H) = ST^H_H(f^H)$. This essentially follows from the computation of Kottwitz in [40, §7].

The following is a special case of the main result of [32].

Theorem 6.17.3. *Keep the setup of Theorem 1.7.2. For p a prime not in Σ and for $a \in \mathbb{N}$ sufficiently divisible, we have*

$$\mathrm{Tr}(\mathrm{Frob}_p^a \times f^{p,\infty} 1_{K_p} dg, \mathbf{H}_c^*(\mathrm{Sh}_K, \mathcal{F}_\lambda^K V) \otimes \mathbb{C}) = \sum_{(H,s,\eta) \in \mathscr{E}(G)} \iota(G,H) ST_e^H(f^H).$$

□

Corollary 6.17.4. *Keep the setup of Theorem 1.7.2. In the even case assume $\mathbb{V}$ is a formal sum $\mathbb{V} = \mathbb{V}_1 + {}^{\mathrm{a}}\mathbb{V}_1$, where ${}^{\mathrm{a}}\mathbb{V}_1$ defined in Definition 4.11.12. Let p be a prime not in Σ. For $a \in \mathbb{N}$ sufficiently divisible, we have*

$$\mathrm{Tr}(\mathrm{Frob}_p^a \times f^{p,\infty} 1_{K_p} dg, \mathbf{IH}^*(\overline{\mathrm{Sh}_K},\ \mathbb{V}) \otimes \mathbb{C}) = \sum_{(H,s,\eta) \in \mathscr{E}(G)} \iota(G,H) ST^H(f^H).$$

Proof. This follows from Theorem 1.7.2, Corollary 6.17.1, and Theorem 6.17.3. □

References

[1] J. Arthur. The L^2-Lefschetz numbers of Hecke operators. *Invent. Math.*, 97(2):257–290, 1989.

[2] J. Arthur. A stable trace formula. II. Global descent. *Invent. Math.*, 143(1):157–220, 2001.

[3] J. Arthur. A stable trace formula. I. General expansions. *J. Inst. Math. Jussieu*, 1(2):175–277, 2002.

[4] J. Arthur. A stable trace formula. III. Proof of the main theorems. *Ann. of Math. (2)*, 158(3):769–873, 2003.

[5] J. Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[6] J. Arthur and L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[7] A. Borel. Automorphic L-functions. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[8] M. Borovoi. Abelian Galois cohomology of reductive groups. *Mem. Amer. Math. Soc.*, 132(626):viii+50, 1998.

[9] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

[10] L. Clozel. The fundamental lemma for stable base change. *Duke Math. J.*, 61(1):255–302, 1990.

[11] L. Clozel and P. Delorme. Pseudo-coefficients et cohomologie des groupes de Lie réductifs réels. *C. R. Acad. Sci. Paris Sér. I Math.*, 300(12):385–387, 1985.

[12] R. Cluckers and F. Loeser. Constructible exponential functions, motivic Fourier transform and transfer principle. *Ann. of Math. (2)*, 171(2):1011–1065, 2010.

[13] P. Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[14] P. Deligne. *Cohomologie étale.* Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[15] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.

[16] K. Fujiwara. Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture. *Invent. Math.*, 127(3):489–533, 1997.

[17] M. Goresky, G. Harder, and R. MacPherson. Weighted cohomology. *Invent. Math.*, 116(1-3):139–213, 1994.

[18] M. Goresky, R. Kottwitz, and R. MacPherson. Discrete series characters and the Lefschetz formula for Hecke operators. *Duke Math. J.*, 89(3):477–554, 1997.

[19] T. J. Haines and S. Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. *Represent. Theory*, 14:264–284, 2010.

[20] T. C. Hales. A simple definition of transfer factors for unramified groups. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 109–134. Amer. Math. Soc., Providence, RI, 1993.

[21] T. C. Hales. On the fundamental lemma for standard endoscopy: reduction to unit elements. *Canad. J. Math.*, 47(5):974–994, 1995.

[22] Harish-Chandra. Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. *Acta Math.*, 113:241–318, 1965.

[23] R. A. Herb. Characters of averaged discrete series on semisimple real Lie groups. *Pacific J. Math.*, 80(1):169–177, 1979.

[24] F. Hörmann. *The geometric and arithmetic volume of Shimura varieties of orthogonal type*, volume 35 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2014.

[25] C. Johansson. A remark on a conjecture of Buzzard-Gee and the cohomology of Shimura varieties. *Math. Res. Lett.*, 20(2):279–288, 2013.

[26] T. Kaletha. Endoscopic character identities for depth-zero supercuspidal L-packets. *Duke Math. J.*, 158(2):161–224, 2011.

[27] T. Kaletha. Rigid inner forms of real and p-adic groups. *Ann. of Math. (2)*, 184(2):559–632, 2016.

[28] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White. Endoscopic Classification of Representations: Inner Forms of Unitary Groups. *ArXiv e-prints*, Sept. 2014.

[29] M. Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.

[30] M. Kisin. Mod p points on Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 30(3):819–914, 2017.

[31] M. Kisin and G. Pappas. Integral models of Shimura varieties with parahoric level structure. *ArXiv e-prints*, Dec. 2015.

[32] M. Kisin, S. W. Shin, and Y. Zhu. The stable trace formula for certain Shimura varieties of abelian type. in preparation.

[33] B. Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

[34] B. Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

[35] R. Kottwitz and D. Shelstad. On Splitting Invariants and Sign Conventions in Endoscopic Transfer. *ArXiv e-prints*, Jan. 2012.

[36] R. E. Kottwitz. Shimura varieties and twisted orbital integrals. *Math. Ann.*, 269(3):287–300, 1984.

[37] R. E. Kottwitz. Stable trace formula: cuspidal tempered terms. *Duke Math. J.*, 51(3):611–650, 1984.

[38] R. E. Kottwitz. Stable trace formula: elliptic singular terms. *Math. Ann.*, 275(3):365–399, 1986.

[39] R. E. Kottwitz. Tamagawa numbers. *Ann. of Math. (2)*, 127(3):629–646, 1988.

[40] R. E. Kottwitz. Shimura varieties and λ-adic representations. In *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 161–209. Academic Press, Boston, MA, 1990.

[41] R. E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.

[42] R. E. Kottwitz and D. Shelstad. Foundations of twisted endoscopy. *Astérisque*, (255):vi+190, 1999.

[43] J.-P. Labesse. Cohomologie, stabilisation et changement de base. *Astérisque*, (257):vi+161, 1999. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[44] J.-P. Labesse. Stable twisted trace formula: elliptic terms. *J. Inst. Math. Jussieu*, 3(4):473–530, 2004.

[45] J.-P. Labesse. Introduction to endoscopy: Snowbird lectures, revised version, May 2010 [revision of mr2454335]. In *On the stabilization of the trace formula*, volume 1 of *Stab. Trace Formula Shimura Var. Arith. Appl.*, pages 49–91. Int. Press, Somerville, MA, 2011.

[46] K.-W. Lan. *Arithmetic compactifications of PEL-type Shimura varieties*, volume 36 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2013.

[47] K.-W. Lan and B. Stroh. Nearby cycles of automorphic étale sheaves, II. preprint, `http://www-users.math.umn.edu/~kwlan/articles/bonn-2016.pdf`.

[48] R. Langlands and D. Shelstad. Descent for transfer factors. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 485–563. Birkhäuser Boston, Boston, MA, 1990.

[49] R. P. Langlands. Shimura varieties and the Selberg trace formula. *Canad. J. Math.*, 29(6):1292–1299, 1977.

[50] R. P. Langlands. Automorphic representations, Shimura varieties, and motives. Ein Märchen. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 205–246. Amer. Math. Soc., Providence, R.I., 1979.

[51] R. P. Langlands. On the zeta functions of some simple Shimura varieties. *Canad. J. Math.*, 31(6):1121–1216, 1979.

[52] R. P. Langlands. *Base change for* GL(2), volume 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[53] R. P. Langlands and D. Shelstad. On the definition of transfer factors. *Math. Ann.*, 278(1-4):219–271, 1987.

[54] K. Madapusi Pera. Toroidal compactifications of integral models of Shimura varieties of Hodge type. *ArXiv e-prints*, Nov. 2012.

[55] K. Madapusi Pera. Integral canonical models for spin Shimura varieties. *Compos. Math.*, 152(4):769–824, 2016.

[56] S. Morel. *Complexes d'intersection des compactifications de Baily-Borel. Le cas des groupes unitaires sur Q.* PhD thesis, PhD Thesis, 2006, 2006.

[57] S. Morel. Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel. *J. Amer. Math. Soc.*, 21(1):23–61, 2008.

[58] S. Morel. The intersection complex as a weight truncation and an application to Shimura varieties. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 312–334, New Delhi, 2010. Hindustan Book Agency.

[59] S. Morel. *On the cohomology of certain noncompact Shimura varieties*, volume 173 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2010. With an appendix by Robert Kottwitz.

[60] S. Morel. Cohomologie d'intersection des variétés modulaires de Siegel, suite. *Compos. Math.*, 147(6):1671–1740, 2011.

[61] B. C. Ngô. Le lemme fondamental pour les algèbres de Lie. *Publ. Math. Inst. Hautes Études Sci.*, (111):1–169, 2010.

[62] R. Pink. *Arithmetical compactification of mixed Shimura varieties*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 209. Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.

[63] R. Pink. On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. *Math. Ann.*, 292(2):197–240, 1992.

[64] R. Pink. On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne. *Ann. of Math. (2)*, 135(3):483–525, 1992.

[65] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[66] D. Shelstad. Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure. *Represent. Theory*, 12:369–402, 2008.

[67] D. Shelstad. A note on real endoscopic transfer and pseudo-coefficients. 2010. preprint, `http://andromeda.rutgers.edu/~shelstad/shelstadnotepseudocoeffs.pdf`.

[68] D. Shelstad. Tempered endoscopy for real groups. II. Spectral transfer factors. In *Automorphic forms and the Langlands program*, volume 9 of *Adv. Lect. Math. (ALM)*, pages 236–276. Int. Press, Somerville, MA, 2010.

[69] D. Shelstad. On the structure of endoscopic transfer factors. *ArXiv e-prints*, Dec. 2014.

[70] D. Shelstad. On elliptic factors in real endoscopic transfer I. In *Representations of reductive groups*, volume 312 of *Progr. Math.*, pages 455–504. Birkhäuser/Springer, Cham, 2015.

[71] S. W. Shin and N. Templier. Sato-Tate theorem for families and low-lying zeros of automorphic L-functions. *Invent. Math.*, 203(1):1–177, 2016. Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok.

[72] O. Taïbi. Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula. *ArXiv e-prints*, June 2014.

[73] O. Taïbi. Arthur's multiplicity formula for certain inner forms of special orthogonal and symplectic groups. *ArXiv e-prints*, Oct. 2015.

[74] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

[75] G. van Dijk. Computation of certain induced characters of p-adic groups. *Math. Ann.*, 199:229–240, 1972.

[76] Y. Varshavsky. A proof of a generalization of Deligne's conjecture. *Electron. Res. Announc. Amer. Math. Soc.*, 11:78–88, 2005.

[77] D. A. Vogan, Jr. Gel′fand-Kirillov dimension for Harish-Chandra modules. *Invent. Math.*, 48(1):75–98, 1978.

[78] D. A. Vogan, Jr. The local Langlands conjecture. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 305–379. Amer. Math. Soc., Providence, RI, 1993.

[79] J.-L. Waldspurger. Le lemme fondamental implique le transfert. *Compositio Math.*, 105(2):153–236, 1997.

[80] J.-L. Waldspurger. Endoscopie et changement de caractéristique. *J. Inst. Math. Jussieu*, 5(3):423–525, 2006.

[81] J.-L. Waldspurger. Les facteurs de transfert pour les groupes classiques: un formulaire. *Manuscripta Math.*, 133(1-2):41–82, 2010.

[82] J. Wildeshaus. Mixed sheaves on Shimura varieties and their higher direct images in toroidal compactifications. *J. Algebraic Geom.*, 9(2):323–353, 2000.